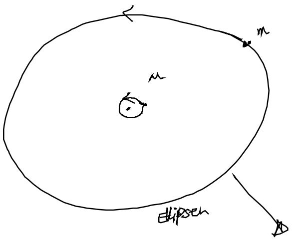
– Beispiele f. Lagrang gl. 2 Art:

Euler-Lagrange: $\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0$ Anstatt $F_i = m\ddot{q}_i$

$$L = T - V$$

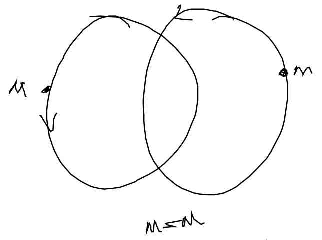
a) Keplerproblem: besonderes Beispiel von Zweikörpersysteme:

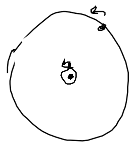


$F \propto \frac{1}{r^2}$

$m \ll M$

S.M → punkt





$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\right) + \frac{GMm}{r}$$

↳ $\vec{r} = r\,\hat{e}_r \Rightarrow \dot{\vec{r}} = \dot{r}\,\hat{e}_r + r\sin\theta\,\dot{\phi}\,\hat{e}_\phi + r\dot{\theta}\,\hat{e}_\theta$

$\hat{e}_r(\theta,\phi)$

$q_i = \{r, \theta, \phi\}$

$r:\ \frac{\partial L}{\partial r} = m r\dot{\theta}^2 + m r\sin^2\theta\,\dot{\phi}^2 - \frac{GMm}{r^2}$, $p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$ $\left(p_i = \frac{\partial L}{\partial \dot{q}_i}\right)$

$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m r^2\dot{\theta}$

$\theta:\ \frac{\partial L}{\partial \theta} = m r^2 \sin\theta\cos\theta\,\dot{\phi}^2$,

$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m r^2 \sin^2\theta\,\dot{\phi} = \text{const!}$

$\phi:\ \frac{\partial L}{\partial \phi} = 0$,

$$E = H = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L$$

$$= \dot{r}\,m\dot{r} + \dot{\phi}\,m r^2\sin^2\theta\,\dot{\phi} + \dot{\theta}\,m r^2\dot{\theta}$$

$$- \frac{m\dot{r}^2}{2} - \frac{m r^2}{2}\sin^2\theta\,\dot{\phi}^2 - \frac{m}{2}r^2\dot{\theta}^2 + V_G$$

$$= \frac{m}{2}\dot{r}^2 + \frac{m r^2}{2}\sin^2\theta\,\dot{\phi} + \frac{m}{2} r^2\dot{\theta}^2 + V_G$$

DrehImpuls $\underline{L_z} = m\vec{r}\times\vec{p} = m(x\dot{y} - \dot{x}y)$; $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$

$$L_z = m r\sin\theta\cos\phi\left(\dot{r}\sin\theta\sin\phi + r\cos\theta\,\dot{\theta}\sin\phi + r\sin\theta\cos\phi\,\dot{\phi}\right)$$

$$- r\sin\theta\sin\theta\left(\dot{r}\sin\theta\cos\phi + r\cos\theta\,\dot{\theta}\cos\phi - r\sin\theta\sin\phi\,\dot{\phi}\right)$$

$L_z = m r^2\dot{\phi}\sin^2\theta$; $\theta = \frac{\pi}{2}$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot\theta} = \frac{\partial L}{\partial\theta} \quad \begin{cases} \frac{d}{dt}(mr\dot\theta) = mr^2\sin\theta\cos\theta\dot\phi^2 \\ (\dot r^2)\dot\theta + r^2\ddot\theta = r^2\sin\theta\cos\theta\dot\phi^2 \end{cases}$$

wenn

$\theta = \frac{\pi}{2}$ Bewegung, so ist $\theta = \frac{\pi}{2}$ $\forall$ Zeite!

Bewegungsgleichung: $\frac{d}{dt}\frac{\partial L}{\partial\dot r} = \frac{\partial L}{\partial r}$ $\quad \dot\theta = 0$, $\theta = \frac{\pi}{2} \to \sin\theta = 1$

$$\ddot r = r\dot\phi^2 - \frac{GM}{r^2}$$

$$m\ddot r = \underbrace{mr\dot\phi^2}_{\downarrow} - \frac{GmM}{r^2} \quad ; \quad \dot\phi = \left(\frac{L_2}{mr^2}\right)^2$$

$$\ddot r = \frac{L_2^2}{m^2r^3} - \frac{GM}{r^2}$$

$$m\ddot r = -\frac{\partial V}{\partial r} \Rightarrow V_{eff} = \frac{L_2^2}{2mr^2} - \frac{GmM}{r}$$

$$X \begin{cases} m\ddot x_1 = F_{12} \\ m\ddot x_2 = F_{21} \\ r = x_1 - x_2 \Rightarrow \text{---} \end{cases}$$

b) Teilchen in elektromagnetischem Feld

$$\Rightarrow \vec F_2 = q\left(\vec E(\vec r, t) + \vec v(t) \times \vec B(\vec r, t)\right)$$

wir können zunächst kein Potential f. die Kraft, E, B auswählen! $\underline{\underline{V?}}$

$$\boxed{E = -\nabla\phi - \partial_t A_i ; B = \vec\nabla\times\vec A}$$

el. Potential; Vektorpotential

$\to$ $L = T - V$

unklar was V ist?!

Aber kann man L irgendwie ergeben, Hauptsache die richtige Bewegungsgleichung entsteht!

Feynman: "The question of what the action ($S = \int dt \sim$) should be, must be determined by some kind of trial and error, ... you have to fiddle around"

$$L_{Teil} = T = \frac{m\dot{\vec r}^2}{2}$$

$$L_{Feld} = \frac{\varepsilon_0}{2}\vec E^2 - \frac{1}{2\mu_0}\vec B^2 \mapsto \text{später}$$

$$\Rightarrow L_{\text{Teilchen-Feld}} = \underbrace{-q\,\phi(\vec r, t)}_{\text{Skalar}} + q\,\underbrace{\dot{\vec r}\cdot\vec A(\vec r, t)}_{\text{Skalar}} = \mathcal{L}(\vec r, \dot{\vec r}, t)$$

Vektor

$V = q\phi$

$$\frac{\partial L}{\partial x_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot x_i}$$

(i) $\frac{\partial L}{\partial x} = -q\,\partial_x\phi + q\,\dot{\vec r}\cdot(\partial_x A_x, \partial_y A_y, \partial_z A_z)$

$= -q\,\partial_x\phi + q\,(\dot x\,\partial_x A_x + \dot y\,\partial_x A_y + \dot z\,\partial_x A_z$

$-(-\dot z\,\partial_x A_z) + \dot z\,\partial_z A_z + \dot y\,\partial_y A_x + \dot y\,\partial_x A_y)$

$= -q\,\partial_x\phi + q\,(\dot{\vec r}\cdot\nabla)\,A_x + q\,[\dot{\vec r}\times(\nabla\times A)]_x$

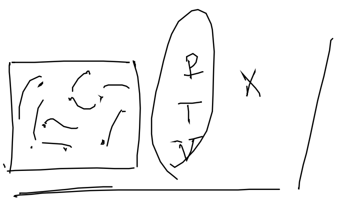
$= -q\,\partial_x\phi + q\,(\frac{d}{dt}A_x - \frac{\partial}{\partial t}A_x) + q\,[\dot{\vec r}\times(\nabla\times A)]$

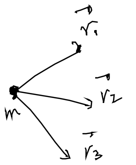
(ii) $\frac{d}{dt}\frac{\partial L}{\partial \dot x} = \frac{d}{dt}(m\dot x) + q\,\frac{d}{dt}(A_x(r,t)) = m\ddot x + q\,\frac{d}{dt}A_x$

$m\ddot x = -q\,\partial_x\phi - q\,\frac{\partial}{\partial t}A + q\,[\dot{\vec r}\times B]_x$

$F_x \Rightarrow m\ddot x = q\,\{E_x(\vec r,t) + [\dot{\vec r}\times B(\vec r,t)]\}_x$

2.4 Euler-Lagrangegleichung f. Vielteilchensystem.





N Teilchen $\vec r_i, \vec r_1, \ldots \vec r_N$

Def. der L.g. $L = T(\{\dot x_i\}) - V\{x_i\}_{i=1\to N}$

$\{x_i\}\to\{q_i\}$: $L = T(\dot q_i) - V(q_i)$ keine Annahme über „i" gemacht!

daher Ableitungen f Teilchen aus Hamiltonpri / PKW

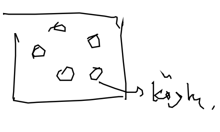
$$\frac{d}{dt}\frac{\partial L}{\partial \dot q_i} = \frac{\partial L}{\partial q_i}$$

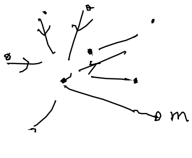
Beispiel Gravitation ww von Punktmasse

a) potentielle Energie einer Massenverteilung

Ansammlung von Punktmassen,

Seien die Teilchen ruhend. z.B.





$E = T + V$,

$\sum_i \frac{m_i}{2} \dot{r}_i^2 + V\{r_i\} = V\{r_i\}$ $\rightarrow$ es verbindet V zu bestimmen!

$\downarrow$ 0

– Das Teilchen im ∞, hat keine WW mit beliebiges Teilchen!

wir haben ein Teilchen in Position r_i, bringen wir anders von (∞)

berechnen wir Energiedifferenz zwischen Anfang und Ende Konfigurations! $V_\infty = 0$

– erste Mass ; $\Delta E = 0$, $V_\infty = 0$

– zweite Masse ; m_1 bleibt da $\Rightarrow \Delta E = \Delta V_1 = -\frac{G m_1 m_2}{|\vec{r}_1 - \vec{r}_2|}$

– dritte Masse ; m_1, m_2 bleiben da $\rightarrow$ $\Delta E = \Delta V_2 = -\frac{G m_1 m_3}{|\vec{r}_1 - \vec{r}_3|} - \frac{G m_2 m_3}{|\vec{r}_2 - \vec{r}_3|}$

$\vdots$

$$E = V = -\sum_{u=1}^{N} \sum_{i<u} \frac{G m_u m_i}{|\vec{r}_i - \vec{r}_u|} = \frac{1}{2} \sum_{u,i} \frac{G m_i m_u}{|r_i - r_u|}$$

b) Lagrangefunktion der WW Teilchen ;

$$L = T - V = \sum_{u=1}^{N} \frac{m}{2} \dot{\vec{r}}_u^2 + \frac{1}{2} \sum_{m=1}^{N} \sum_{i=1}^{N} \frac{G m_u m_i}{|\vec{r}_m - \vec{r}_i|} \quad *$$

$$\underline{*} \quad \frac{\partial L}{\partial \dot{x}_{ej}} = \frac{\partial}{\partial \dot{x}_{ej}} \sum_{u,i} \frac{m_u \dot{x}_{ui}^2}{2} = \sum_{ui} \frac{m_u}{2} \frac{\partial}{\partial \dot{x}_{ej}} \dot{x}_{ui}^2$$

$$= \sum_{ui} 2 \dot{x}_{ui} \frac{\partial \dot{x}_{ui}}{\partial \dot{x}_{ej}} = \sum_{ui} m_u \dot{x}_{ui} \delta_{ue} \delta_{ij} = m_e \dot{x}_{ej}$$

$$* \quad \frac{\partial L}{\partial x_{ej}} = \frac{\partial}{\partial x_{ej}} \frac{1}{2} \sum_{u,m} \frac{G m_u m_m}{|\vec{r}_u - \vec{r}_m|} = -\sum_m \frac{G m_e m_n (x_{ej} - x_{nj})}{|\vec{r}_e - \vec{r}_u|^3}$$

$\times$

$$m_e \ddot{x}_{ej} = -G m_e \sum_u \frac{(x_{ej} - x_{uj})}{|r_e - r_u|^3}$$