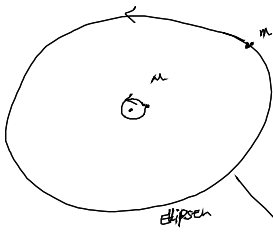


- Beispiele f. Lagrang. 2. Art:

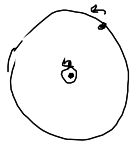
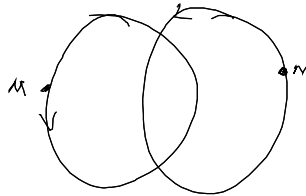
Euler-Lagrange: $\frac{\partial L}{\partial \dot{q}_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = 0$ Ansatz $F_i = m \ddot{q}_i$

$L = T - V$

a) Keplersproblem: interessantes Beispiel von Zweikörpersysteme:



$F \propto \frac{1}{r^2}$
 $m \ll M$
 d.M. \rightarrow Punkt



$L = \frac{m}{2} (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2) + \frac{G M m}{r}$

$\vec{r} = r \vec{e}_r \Rightarrow \dot{\vec{r}} = \dot{r} \vec{e}_r + r \sin \theta \dot{\phi} \hat{\phi} + r \dot{\theta} \hat{\theta}$
 $\hat{e}_r(\theta, \phi)$

$q_i = \{r, \theta, \phi\}$

$r: \frac{\partial L}{\partial r} = m r \dot{\theta}^2 + m r \sin^2 \theta \dot{\phi}^2 - \frac{G M m}{r^2}$, $p_r = \frac{\partial L}{\partial \dot{r}} = m \dot{r}$ ($p_i = \frac{\partial L}{\partial \dot{q}_i}$)

$\theta: \frac{\partial L}{\partial \theta} = m r^2 \sin \theta \cos \theta \dot{\phi}^2$,

$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta}$

$\phi: \frac{\partial L}{\partial \phi} = 0$,

$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m r^2 \sin^2 \theta \dot{\phi} = \text{const!}$

$E = H = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L$

$= \dot{r} m \dot{r} + \dot{\theta} m r^2 \sin^2 \theta \dot{\phi} + \dot{\phi} m r^2 \dot{\phi}$

$= \frac{m \dot{r}^2}{2} + \frac{m r^2}{2} \sin^2 \theta \dot{\phi}^2 - \frac{m}{2} r^2 \dot{\theta}^2 + V_G$

$= \frac{m}{2} \dot{r}^2 + \frac{m r^2}{2} \sin^2 \theta \dot{\phi}^2 + \frac{m}{2} r^2 \dot{\theta}^2 + V_G$

Drehimpuls $L_z = m \vec{r} \times \dot{\vec{r}} = m (x \dot{y} - \dot{x} y)$; $x = r \sin \theta \cos \phi$
 $y = r \sin \theta \sin \phi$

$L_z = m r \sin \theta \cos \phi (\dot{r} \sin \theta \sin \phi + r \cos \theta \dot{\theta} \sin \phi + r \sin \theta \cos \theta \dot{\phi})$

$- r \sin \theta \sin \phi (\dot{r} \sin \theta \cos \phi + r \cos \theta \dot{\theta} \cos \phi - r \sin \theta \sin \theta \dot{\phi})$

$L_z = m r^2 \dot{\phi} \sin^2 \theta$; $\theta = \frac{\pi}{2}$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta} \quad \left\{ \begin{array}{l} \frac{d}{dt} (m r \dot{\theta}) = m r^2 \sin \theta \omega \dot{\phi} \\ (\dot{r}) \dot{\theta} + r^2 \ddot{\theta} = r^2 \sin \theta \omega \dot{\phi}^2 \end{array} \right.$$

wenn $\theta = \frac{\pi}{2}$ Bewegung ist ist $\theta = \frac{\pi}{2} \forall$ Zeit!

Bewegungsgleichung: $\frac{d}{dt} \frac{\partial L}{\partial \dot{r}} = \frac{\partial L}{\partial r} \quad \dot{\theta} = 0, \theta = \frac{\pi}{2} \rightarrow \sin \theta = 1$

$$\ddot{r} = r \dot{\phi}^2 - \frac{GM}{r^2}$$

$$m \ddot{r} = \underbrace{m r \dot{\phi}^2}_{\downarrow} - \frac{GMm}{r^2} \quad ; \quad \dot{\phi} = \left(\frac{L_z}{m r^2} \right)^2$$

$$\ddot{r} = \frac{L_z^2}{m^2 r^3} - \frac{GM}{r^2}$$

$$m \ddot{r} = -\frac{\gamma V}{\partial r} \Rightarrow V_{\text{eff}} = \frac{L_z^2}{2 m r^2} - \frac{GMm}{r}$$

$$X \left\{ \begin{array}{l} m \ddot{x}_1 = F_{12} \\ m \ddot{x}_2 = F_{21} \end{array} \right\}$$

$V = x_1 - x_2 \Rightarrow$

b) Teilchen in elektromagnetischem Feld

$$\vec{f} = q (\vec{E}(\vec{r}, t) + \vec{v}(t) \times \vec{B}(\vec{r}, t))$$

wie können wir nicht kein potential f. die kraft, E, B auswählen! V?

$$\boxed{E = -\nabla \phi - \frac{\partial A}{\partial t}; \quad B = \nabla \times A}$$

\downarrow elektr. potential \downarrow vektorpotential

$$L = T - V$$

klar was V ist?!

Aber kann man L irgendwie ergreifen, Hauptsache die richtige Bewegungsgleichung entsteht!

Feynman: 'The question of what the action ($S = \int dt$) should be, must be determined by some kind of trial and error, ... you have to fiddle around'

$$L_{\text{Teil}} = T = \frac{m \dot{\vec{r}}^2}{2}$$

$$L_{\text{feld}} = \frac{\epsilon_0}{2} \vec{E}^2 - \frac{1}{2\mu_0} \vec{B}^2 \quad \rightarrow \text{später}$$

$$\Rightarrow L_{\text{Teilchen-feld}} = -q \phi(\vec{r}, t) + q \dot{\vec{r}} \cdot \vec{A}(\vec{r}, t) = L(\vec{r}, \dot{\vec{r}}, t)$$

$\underbrace{-q \phi}_{\text{Skalar}} + \underbrace{q \dot{\vec{r}} \cdot \vec{A}}_{\text{Skalar}} \rightarrow \text{Skalar}$

$V = q \phi$

$$\frac{\partial L}{\partial x_i} = \frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i}$$

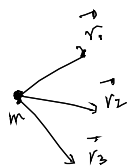
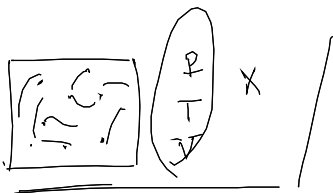
$$\begin{aligned} \text{(i)} \quad \frac{\partial L}{\partial \mathbf{x}} &= -q \nabla \phi + q \dot{\mathbf{r}} \cdot (\partial_x A_x, \partial_y A_y, \partial_z A_z) \\ &= -q \nabla \phi + q (\dot{x} \partial_x A_x + \dot{y} \partial_y A_y + \dot{z} \partial_z A_z) \\ &\quad - (-\dot{z} \partial_x A_z) + \dot{z} \partial_z A_z + \dot{y} \partial_y A_y + \dot{y} \partial_x A_x \\ &= -q \nabla \phi + q (\dot{\mathbf{r}} \cdot \nabla) A_x + q [\dot{\mathbf{r}} \times (\nabla \times A)]_x \\ &= -q \nabla \phi + q \left(\frac{d}{dt} A_x - \frac{\partial}{\partial t} A_x \right) + q [\dot{\mathbf{r}} \times (\nabla \times A)]_x \end{aligned}$$

$$\text{(ii)} \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{\mathbf{x}}} = \frac{d}{dt} (m \dot{\mathbf{x}}) + q \frac{d}{dt} (A_x(v(t))) = m \ddot{\mathbf{x}} + q \frac{d}{dt} A_x$$

$$m \ddot{\mathbf{x}} = -q \nabla \phi - q \frac{\partial}{\partial t} A + q [\dot{\mathbf{r}} \times \mathbf{B}]_x$$

$$\mathbf{F}_x \Rightarrow m \ddot{\mathbf{x}} = q \left(\mathbf{E}_x(\dot{\mathbf{r}}, t) + [\dot{\mathbf{r}} \times \mathbf{B}(\dot{\mathbf{r}}, t)]_x \right)$$

2.4 Euler-Lagrange-Gleichung f. Vielteilchen system.



N Teilchen $\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N$

$$\text{Def. der Lg } L = T(\{\dot{x}_i\}) - V(\{x_i\}) \quad i=1 \rightarrow N$$

$$\{x_i\} \rightarrow \{q_i\} = L = T(q_i) - V(q_i)$$

daher Ableitungen f. Teilchen aus Hamiltonprin / p.k.w. keine Annahme über „i“ gemacht!

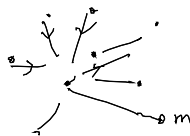
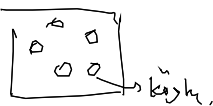
$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}$$

Beispiel Charakterisierung von punktmasse

a) potentielle Energie eine Massverteilung

Aussammlung von punktmassen,

Seien die Teilchen reihend - z.B.



... ..

$E = T + V$
 $\sum_i \frac{m_i}{2} \dot{r}_i^2 + V\{r_i\} = V\{r_i\} \rightarrow$ es verbleibt V zu bestimmen!

- Bei Teilchen im ∞ , hat keine Wkt mit dieses Teilchen!

wir haben ein Teilchen in position r_i , bringen wir andere von ∞ |
 berechnen Energie differenz zwischen anfang und ende konfigurationsraum | $V = \infty$
 wir

- erste mass ; $\Delta E = 0$, $V_1 = 0$

- zweite masse ; m_1 bleibt da $\Rightarrow \Delta E = \Delta V_1 = -\frac{G m_1 m_2}{|\vec{r}_1 - \vec{r}_2|}$

- dritte masse ; m_1, m_2 bleiben da $\rightarrow \Delta E = \Delta V_2 = -\frac{G m_1 m_3}{|\vec{r}_1 - \vec{r}_3|} - \frac{G m_2 m_3}{|\vec{r}_2 - \vec{r}_3|}$

$$E = V = -\sum_{\alpha=1}^N \sum_{i \neq \alpha} \frac{G m_\alpha m_i}{|\vec{r}_i - \vec{r}_\alpha|} = \frac{1}{2} \sum_{i,j} \frac{G m_i m_j}{|\vec{r}_i - \vec{r}_j|}$$

b) Lagrange funktion der n teilchen ;

$$L = T - V = \sum_{\alpha=1}^n \frac{m_\alpha}{2} \dot{r}_\alpha^2 + \frac{1}{2} \sum_{m \neq i} \sum_{l \neq i} \frac{G m_m m_l}{|\vec{r}_m - \vec{r}_l|} \quad *$$

$$* \frac{\delta L}{\delta x_{ij}} = \frac{\delta}{\delta x_{ij}} \sum_{\alpha} \frac{m_\alpha}{2} \dot{x}_{\alpha i}^2 = \sum_{\alpha} \frac{m_\alpha}{2} \frac{\delta}{\delta x_{ij}} \dot{x}_{\alpha i}^2$$

$$= \sum_{\alpha} 2 \dot{x}_{\alpha i} \frac{\delta \dot{x}_{\alpha i}}{\delta x_{ij}} = \sum_{\alpha} m_\alpha \dot{x}_{\alpha i} \delta_{\alpha i j} = m_\alpha \dot{x}_{ij}$$

$$* \frac{\delta L}{\delta x_{ij}} = \frac{\delta}{\delta x_{ij}} \frac{1}{2} \sum_{\alpha \neq m} \frac{G m_\alpha m_m}{|\vec{r}_\alpha - \vec{r}_m|} = - \sum_m \frac{G m_\alpha m_m (x_{ej} - x_{ij})}{|\vec{r}_\alpha - \vec{r}_m|^3}$$

$$m_\alpha \ddot{x}_{ej} = - G m_\alpha \sum_m \frac{(x_{ej} - x_{ij})}{|\vec{r}_\alpha - \vec{r}_m|^3}$$