Wdh

- Feldenergie einer PL divergiert
- El. Feld ist an geol. GF unstetig

Gauss: $\underline{n}\cdot(\underline{E}_a - \underline{E}_i) = 4\pi\sigma$

Normalen-Vektor des GF; Feld außen; Feld innen; Flächenladungsdichte

Stokes $\underline{t}\cdot(\underline{E}_a - \underline{E}_i) = 0$

Tang.-komp. stetig

- Plattenkondensator

Kap. $C = \frac{q}{U}$

Energie $W = \frac{1}{2} C\cdot U^2$

- Green'sche Theoreme

φ ψ 2 stetig diff'bare sk. Felder auf $V, \partial V$

1.GT: $\int_V [\varphi \Delta\psi + (\nabla\varphi)(\nabla\psi)] d^3r = \oint_{\partial V} \varphi \frac{\partial\psi}{\partial n} dS$

$$\frac{\partial\psi}{\partial n} = n\cdot(\nabla\psi)$$

2.GT: $\int_V [\varphi\Delta\psi - \psi\Delta\varphi] d^3r = \oint_{\partial V} \left(\varphi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\varphi}{\partial n}\right) dS$

- Potential als Lsg des RWP

$$\Phi(\underline{r}\in V) = \int_V \frac{\rho(r')}{|r-r'|} d^3r' + \frac{1}{4\pi}\oint_{\partial V}\left[\frac{1}{|r-r'|}\frac{\partial\Phi(r')}{\partial n'} - \Phi(r')\frac{\partial}{\partial n'}\frac{1}{|r-r'|}\right] dS'$$

von-Neumann-RB   Dirichlet RB

"Cauchy"-RB

Leiter: Ladungen verschieben sich bei kleinsten ext. Feldern

Ladungstrennung bis E in Leiter verschwindet

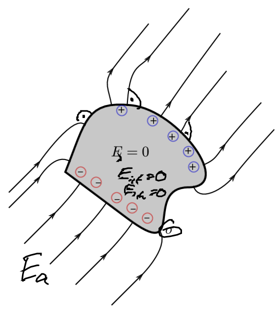
$E|_{\text{Leiter}} = 0 \longrightarrow \Phi|_{\text{Leiter}} = \text{const.}$

$\boxed{\text{Leiteroberflächen sind Äquipotentialflächen}}$

(Dirichlet-RB)

$E_{i,t} = 0 = E_{a,t}$

$E_{a,n} = 4\pi\sigma$ ← "Influenzladung"



1.13. Die Green'sche Funktion der Poisson-Gleichung

allg: Tool zum Lösen von inhomog. linearen part. DGL'n

$$\mathcal{D}_z \, \rho(z) = \varphi(z)$$

Diff.-operator bei Δ — Lösung — Variablen — Inhomogenität z.B $\rho(z)$

vereinfachtes Problem $\mathcal{D}_z \, G(z,s) = \delta(z-s)$

"Green'sche Fkt"

dann $\boxed{\rho(z) = \int G(z,s)\cdot\varphi(s)\,ds}$

$$\mathcal{D}_z \, \rho(z) = \int [\mathcal{D}_z \, G(z,s)]\cdot\varphi(s)\,ds$$

$$= \int \delta(z-s)\,\varphi(s)\,ds = \varphi(z)$$

bei uns: Poisson-GL: $\Delta\Phi = -4\pi\rho$

Green'sche Fkt ist die Lösung für eine Punktladung

$$\Delta G(r,r') = -4\pi\,\delta(r-r')$$

$$G(r,r') = \frac{1}{|r-r'|} \qquad \text{nur eine Lösung}$$

$$G(r,r') = \frac{1}{|r-r'|} + f(r,r') \quad : \quad \Delta f(r,r') = 0$$

Idee Wähle $f(r,r')$ so, dass RB erfüllt sind

Z.G.T.: $$\int_V \left[\Phi(r')\Delta_{r'} G(r,r') - G(r,r')\Delta_{r'}\Phi(r')\right] d^3r'$$

$$= -4\pi\int d^3r'\,\Phi(r')\,\delta(r-r') + 4\pi\int_V d^3r'\,G(r,r')\,\rho(r')$$

$$= \oint_{\partial V}\left[\Phi(r')\frac{\partial G}{\partial n} - G(r,r')\frac{\partial\Phi}{\partial n'}\right] dS'$$

$$\Phi(r) = \int_V \rho(r')\,G(r,r')\,d^3r' - \frac{1}{4\pi}\oint_{\partial V}\left[\Phi(r')\frac{\partial G}{\partial n'} - G(r,r')\frac{\partial\Phi}{\partial n'}\right] dS'$$

a.) Dirichlet RB $\Phi(r')$ für $r'\in\partial V$ gegeben

wollen $\oint_{\partial V} G_D(r,r')\frac{\partial\Phi}{\partial n'}\,dS' = 0$

$\longrightarrow$ wähle $G_D(r,r') = 0 \quad \forall\, r'\in\partial V$

$$\boxed{\Phi(r\in V) = \int_V \rho(r')\,G_D(r,r')\,d^3r' - \frac{1}{4\pi}\oint_{\partial V}\underbrace{\Phi(r')}_{\text{Dirichlet-RB}}\frac{\partial G_D(r,r')}{\partial n'}\,dS'}$$

nützlich

b.) von-Neumann RB

$$\frac{\partial \Phi}{\partial n} = -\underline{E}\cdot\underline{n} \quad \text{sei auf } \partial V \text{ gegeben}$$

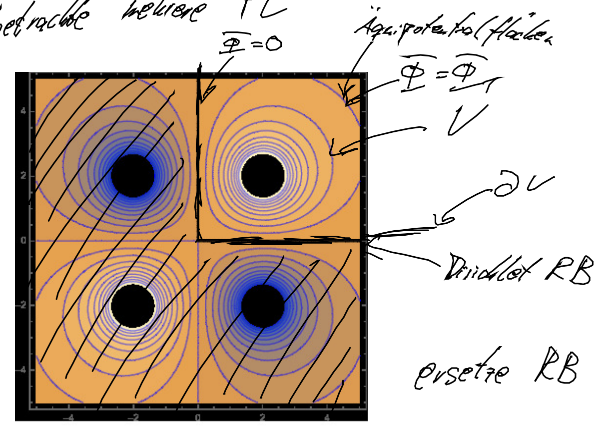
Forderung $\frac{1}{4\pi} \oint_{\partial V} \Phi(r') \frac{\partial G_N}{\partial n'} dS' \stackrel{!}{=} -\Phi_0$

$$\Rightarrow \boxed{\Phi(r\in V) = \Phi_0 + \int \varrho(r')\, G_N(r,r')\, d^3r' + \frac{1}{4\pi}\oint G_N(r,r') \frac{\partial \Phi}{\partial n'} dS'}$$

Wie bekommt man $G_{N/D}$?

1.14. Methode der Bildladungen

betrachte mehrere PL



ersetze RB durch imaginäre Ladungen

Bsp: $V = \{(x,y,z)^T : z > 0\}$

Dirichlet-RB $\Phi(x,y,0) = 0$

$$\lim_{x\to\pm\infty} \Phi(x,y,z) = \lim_{y\to\pm\infty} \Phi(x,y,z) = 0$$

$$\lim_{z\to\infty} \Phi(x,y,z) = 0$$

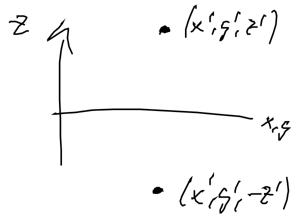
$$G(r,r') = \frac{1}{|r-r'|} + f(r,r') \qquad \Delta_r f(r,r') = 0 \quad \forall r\in V$$

Ort der PL

$$\oint G_D(r,r') \frac{\partial \Phi}{\partial n'} dS' = 0$$

$$\lim_{z'\to 0} f(r,r') = \frac{-1}{\sqrt{(x-x')^2+(y-y')^2+z^2}} \qquad \Rightarrow G_D(r,r') = 0 \quad \forall r' \in \partial V$$

$$\Rightarrow f(r,r') = \frac{-1}{|r-r_B|} = \frac{-1}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}}$$



$$G_D(r,r') = \frac{1}{|r-r'|} - \frac{1}{|r-r_B'|} \qquad \begin{pmatrix} x' \\ y' \\ -z' \end{pmatrix}$$

$$\Delta f(r,r') \propto \delta(r-r_B') = 0 \quad \forall r \in V \quad \text{weil } r_B' \notin V$$

$$G_D(r,r') = \frac{1}{|r-r'|} - \frac{1}{|r-r_B'|}$$

falls $\Phi(x,y,0) \stackrel{!}{=} 0$

$$\Rightarrow \Phi(r\in V) = \int d^3r' \rho(r') \left[\frac{1}{|r-r'|} - \frac{1}{|r-r_B'|}\right]$$

falls $\Phi(x,y,0) \neq 0$

$$\left.\frac{\partial G_D}{\partial z'}\right|_{z'=0} = \underset{\uparrow}{-}\frac{\partial}{\partial z'} G_D(r,r') = \frac{-2z}{[(x-x')^2+(y-y')^2+z^2]^{3/2}}$$

$\in V$ weil in neg. z-Richtung

$$\Phi(r) = \int \rho(r')\, G_D(r,r')\, d^3r' + \frac{2z}{4\pi}\iint dx'dy' \frac{\Phi(x',y',0)}{[(x-x')^2+(y-y')^2+z^2]^{3/2}}$$

"Bildladungen ersetzen Randbedingungen"

1.14.1. Beispiel PL vor Ebene

q am Orte $r' = \begin{pmatrix} 0 \\ 0 \\ z' \end{pmatrix}$ $\quad z' > 0$

Ansatz

$$\Phi(r) = \frac{q}{|r-r'|} + \frac{q_B}{|r-r_B'|}$$

$\leftarrow$ unbekannt

Dirichlet-RB

$$\forall x,y \in \mathbb{R}: \frac{q}{\sqrt{x^2+y^2+(z')^2}} + \frac{q_B}{\sqrt{(x-x_B)^2+(y-y_B)^2+z_B^2}} = 0$$

$x_B = 0$

$y_B = 0$

$z_B = -z'$

$q_B = -q$

$$\Phi(r) = q\left[\frac{1}{\sqrt{x^2+y^2+(z-z')^2}} - \frac{1}{\sqrt{x^2+y^2+(z+z')^2}}\right] \quad \forall r \in V$$

$$E(r) = q\left[\frac{1}{[x^2+y^2+(z-z')^2]^{3/2}}\begin{pmatrix} x \\ y \\ z-z' \end{pmatrix} - \frac{1}{[x^2+y^2+(z+z')^2]^{3/2}}\begin{pmatrix} x \\ y \\ z+z' \end{pmatrix}\right]$$

$$E(x,y,0) = \frac{q}{[x^2+y^2+z'^2]^{3/2}}\begin{pmatrix} 0 \\ 0 \\ -2z' \end{pmatrix}$$

vgl. Normalenkomponente von E an GF

Flächenladungsdichte

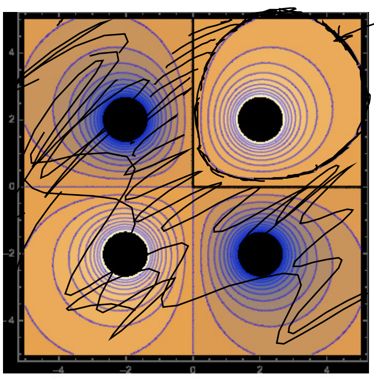
$$\sigma(x,y) = \frac{e_z \cdot E(x,y,0)}{4\pi} = -\frac{q}{2\pi}\,\frac{z'}{[x^2+y^2+(z')^2]^{3/2}}$$

ges. Influenzladung

$$\tilde{q} = \iint dx\,dy\;\sigma(x,y) = -\frac{q}{\cancel{2\pi}}\int_0^\infty d\rho \cancel{\int_0^{2\pi} d\varphi}\;\frac{\rho\cdot z'}{[\rho^2+(z')^2]^{3/2}}$$

$$= -q\cdot z'\int_0^\infty \frac{\rho\,d\rho}{[\rho^2+z'^2]^{3/2}} = -q z'\int_0^\infty d\rho\,\frac{d}{d\rho}\left(-\frac{1}{\sqrt{\rho^2+z'^2}}\right)$$

$$= -q z'\left(-0 + \frac{1}{z'}\right) = -q = +q_B$$



$\Phi = \Phi_1$