Wdh.

+ Hohl-Leiter   transv. Komp. sind bestimmt durch E_z, B_z

$$B_\perp = \frac{i c^2}{\omega^2 - k_z^2 c^2}\left[+ \frac{\omega}{c} e_z \times (\nabla_\perp E_z) + k_z (\nabla_\perp B_z)\right] \quad \omega^2 \neq k_z^2 c^2$$

$$E_\perp = \dots \quad (E_z \leftrightarrow B_z \quad c \to -c)$$

TE-Moden: $E_z = 0 \quad B_z \neq 0 \to \left\{\Delta_\perp + k_z^2 - \frac{\omega^2}{c^2}\right\} B_z = 0$

TM-Moden: $B_z = 0 \quad E_z \neq 0 \to \left\{\Delta_\perp + k_z^2 - \frac{\omega^2}{c^2}\right\} E_z = 0$

TEM-Mode: $E_z = B_z = 0$ geht nur bei $\omega^2 = k_z^2 c^2$

+ Lagrange-Hamilton für Punkt-Teilchen (frei)

- $H(\underline{q}, \underline{p}) = \sqrt{\underline{p}^2 c^2 + m_0^2 c^4}$
- $L(q, \dot{q}) = - m_0 c^2 \sqrt{1 - \frac{\dot{q}^2}{c^2}}$

$\dot{q}_i = \text{const}$

+ nicht rel. Lorentz-Kraft

$$L(q, \dot{q}) = \underbrace{\frac{1}{2} m_0 \cdot \dot{q}^2}_{\text{nichtrel.}} - e \cdot \Phi(q(t), t) + \frac{e}{c} \cdot \underbrace{\dot{q}}_{\text{Geschw.}} \cdot A(\underbrace{q(t)}_{\text{Pos. des PT}}, t)$$

$$\Rightarrow \underbrace{m_0 \cdot \ddot{q}_i}_{\text{nichtrel.}} = e\left(E + \frac{\dot{q}}{c} \times B\right)$$

wichtig Unterschied zwischen kinet. & kanon. Impuls

$$H(q, p) = \frac{1}{2} \frac{\left(p - \frac{e}{c} A\right)^2}{m_0} + e \cdot \Phi$$

$$\boxed{p_{kin} = p - \frac{e}{c} A}$$

Ersetzung $p_{kin} \to p - \frac{e}{c} A$ "minimale Kopplung"

8.2.3. Lorentz-Kraft relativistisch

$$H = \sqrt{\left(p - \frac{e}{c} A\right)^2 c^2 + m_0^2 c^4} + e \cdot \Phi$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \leadsto \underline{p}_{kin} = \underline{p} - \frac{e}{c} \underline{A} = \gamma(\dot{\underline{q}}) \cdot m_0 \cdot \dot{\underline{q}}$$

$$= c^2 \frac{\left(p - \frac{e}{c} A\right)}{\sqrt{\left(p - \frac{e}{c} A\right)^2 c^2 + m_0^2 c^4}}$$

$$\underbrace{\dot{p}_i}_{\text{kanon. Impuls}} = -\frac{\partial H}{\partial q_i} = \frac{d}{dt} p_{i,kin} + \underbrace{\frac{e}{c} \frac{d}{dt} A_i}_{\text{nach rechts}}$$

$$\frac{d}{dt} A_i = \frac{\partial A_i}{\partial t} + \sum_j \frac{\partial A_i}{\partial q_j} \cdot \dot{q}_j$$

$$\frac{d}{dt} p_{i,kin} = \frac{d}{dt}\left(\gamma m_0 \cdot \dot{q}_i\right) = e\left[E_i + [\dot{\underline{q}} \times (\nabla \times A)]_i \frac{1}{c}\right]$$

→ H ist richtig

über Legendre-Trafo bekommen wir L

$L = p \cdot \dot{q} - H$

$$\boxed{L = -\frac{m_0 c^2}{\gamma} - e\Phi + \frac{e}{c} \underline{A} \cdot \dot{\underline{q}}}$$

$$A^\mu = \begin{pmatrix} \Phi \\ A_x \\ A_y \\ A_z \end{pmatrix} \qquad v^\mu = \begin{pmatrix} \gamma \cdot c \\ \gamma \cdot v_x \\ \gamma v_y \\ \gamma v_z \end{pmatrix}$$

$$= + \frac{1}{\gamma} \Big[\underbrace{-m_0 c^2 - \frac{e}{c} v_\mu \cdot A^\mu}_{\text{Lorentz-Skalar}} \Big] \qquad v_\mu = \begin{pmatrix} \gamma c \\ -\gamma v_x \\ -\gamma v_y \\ -\gamma v_z \end{pmatrix}$$

$$S = \int_{t_1}^{t_2} L \, dt = \int_{\tau_1}^{\tau_2} \gamma L \, d\tau$$

S ist ein Lorentz-Skalar

8.3. Lagrange-Hamilton für Felder

$$q_i(t) \longrightarrow u_i(\underline{r}, t)$$
$$\dot{q}_i(t) \longrightarrow \nabla u_i(\underline{r}, t), \; \partial_t u_i(\underline{r}, t)$$
$$L(q_i, \dot{q}_i) \longrightarrow \mathcal{L}(u_i, \partial_\mu u_i)$$

Wie müssen die Euler-Lagrange-Gl. modifiziert werden

8.3.1. Ableitung der ELG

$$S[u] = \int_\Omega d^n x \, \mathcal{L}(\underline{x}, u, \partial_i u_j)$$

3+1 Dimensionen; Feld; Ort & Zeit $\underline{x} = (\underline{r}, t)$

Funktional : Funktion → Zahl

1. Variation des Funktionals S an der Stelle $\underline{u}$ in Richtung $\underline{h}$

$$\delta S_{\underline{u}}[\underline{h}] = \frac{d}{d\varepsilon} S[\underline{u} + \varepsilon \cdot \underline{h}]\Big|_{\varepsilon=0} = \lim_{\varepsilon \to 0} \frac{S[\underline{u} + \varepsilon \cdot \underline{h}] - S[\underline{u}]}{\varepsilon}$$

Hamiltonsches Prinzip $\delta S_u[h] = 0 \quad \forall h : \underline{h(\partial\Omega) = 0}$

in Richtung h_i

$$0 \stackrel{!}{=} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_\Omega d^n x \left[\mathcal{L}(\underline{x}, u_i + \varepsilon h_i, \nabla u_i + \varepsilon \nabla h_i, \ldots) - \mathcal{L}(\underline{x}, u_i, \nabla u_i, \ldots) \right]$$

$$= \int_\Omega d^n x \left[\frac{\partial \mathcal{L}}{\partial u_i} \cdot h_i + \sum_{k=1}^n \frac{\partial \mathcal{L}}{\partial(\partial_k u_i)} \frac{\partial h_i}{\partial x_k} \right]$$

part. integrieren

$$= \int_\Omega d^n x \left[\underbrace{\frac{\partial \mathcal{L}}{\partial u_i} - \sum_{k=1}^n \frac{\partial}{\partial x_k} \frac{\partial \mathcal{L}}{\partial(\partial_k u_i)}} \right] h_i(\underline{x}) \stackrel{!}{=} 0$$

$=0$

ELG für Felder

$$\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial(\partial_t \varphi_i)} + \sum_{k=1}^{n-1}\partial_k \frac{\partial\mathcal{L}}{\partial(\partial_k\varphi_i)} - \frac{\partial\mathcal{L}}{\partial\varphi_i} = 0$$

3+1 D kov. Schreibweise

$$\partial^\mu \frac{\partial\mathcal{L}}{\partial(\partial^\mu\varphi_i)} - \frac{\partial\mathcal{L}}{\partial\varphi_i} = 0$$

Lorentz-Skalar

$$S = \int d^4x\, \mathcal{L} = \int dt\, d^3r\, \mathcal{L}$$

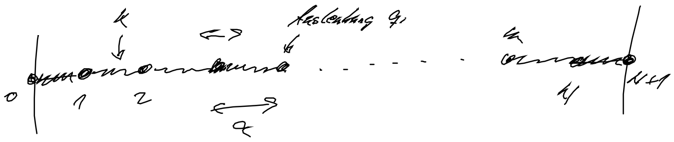
$$\text{LT: } \begin{vmatrix} \gamma & -\gamma\beta & & \\ -\gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{vmatrix} = 1 \quad \text{Lorentz-Skalar} \qquad d^4x = d^4x'$$

$$\tilde{\pi}_i = \frac{\partial\mathcal{L}}{\partial(\partial_t\varphi_i)} \quad \text{kanon. konj. Feldimpuls}$$

$$\text{LT: } \mathcal{H} = \sum_i \tilde{\pi}_i (\partial_t \varphi_i) - \mathcal{L}$$

Dimension einer Energiedichte

8.3.2. Kette von Punktteilchen



$$m\ddot{q}_i = k(q_{i+1} - q_i) - k(q_i - q_{i-1})$$

N identische Massen & N+1 Federn

$$q_0 = q_{N+1} = 0$$

$$L = T - V = \sum_i a\left[\frac{1}{2}\frac{m}{a}\dot{q}_i^2 - \frac{1}{2}ka\left(\frac{q_{i+1}-q_i}{a}\right)^2\right]$$

Limes $\left.\begin{matrix} a \to 0 \\ m \to 0 \\ k \to \infty \end{matrix}\right\}$ $\frac{m}{a} \to \mu$ (1D Massendichte) $\qquad k \cdot a \to Y$ Young-Modul

$$u(x,t) \stackrel{!}{=} q_i(t) \qquad \sum_i a \stackrel{!}{=} \int dx$$

$$\partial_t u(x,t) \stackrel{!}{=} \dot{q}_i(t)$$

$$\partial_x u(x,t) \stackrel{!}{=} \lim_{a\to 0} \frac{q_{i+1}(t) - q_i(t)}{a}$$

$$L = \int_0^{N\cdot a} dx \underbrace{\left[\frac{1}{2}\mu\cdot(\partial_t u)^2 - \frac{1}{2}Y(\partial_x u)^2\right]}_{\mathcal{L}}$$

1+1-dimensional

$$f''(x) = \lim_{a\to 0} \frac{f(x+a) - 2f(x) + f(x-a)}{a^2}$$

$$f'(x) = \lim_{a\to 0} \frac{f(x+a/2) - f(x-a/2)}{a}$$

$$f''(x) = \lim_{a\to 0} \frac{f'(x+\frac{a}{2}) - f'(x-\frac{a}{2})}{a}$$

$$\left(\frac{m}{a}\right)\ddot{q}_i = (k\cdot a)\frac{q_{i+1} - 2q_i + q_{i-1}}{a^2}$$

$$\mu\cdot\partial_t^2 u = Y\cdot\partial_x^2 u \longrightarrow \left\{\partial_x^2 - \frac{\mu}{Y}\cdot\partial_t^2\right\}u(x,t) = 0$$

$$v = \sqrt{\frac{Y}{\mu}} \quad \text{Schallgeschw.}$$

$$0 = \partial_t \frac{\partial\mathcal{L}}{\partial(\partial_t u)} + \partial_x \frac{\partial\mathcal{L}}{\partial(\partial_x u)} - \frac{\partial\mathcal{L}}{\partial u}$$

$$= \mu\cdot\partial_t(\partial_t u) - Y\partial_x(\partial_x u) \longrightarrow \text{reprod. die Wellengl.}$$

in ED 3+1 dimensional
4 Felder Φ, $\underline{A}$

8.3.3. Elektrostatik

$$u_i \longrightarrow \Phi \qquad \underline{E} = -\nabla\Phi$$

$$\boxed{\mathcal{L}_{ES} = \frac{1}{8\pi}(\nabla\Phi)^2 - \rho\cdot\Phi}$$

$$0 = \sum_i \partial_i \frac{\partial\mathcal{L}}{\partial(\partial_i\Phi)} + \partial_t \frac{\partial\mathcal{L}}{\partial(\partial_t\Phi)} - \frac{\partial\mathcal{L}}{\partial\Phi}$$

$$= \sum_i \partial_i \frac{1}{4\pi}(\partial_i\Phi) + \rho \longrightarrow \Delta\Phi = -4\pi\rho \quad \text{Poisson-Gl.}$$

$$\nabla\cdot\underline{E} = 4\pi\rho$$

8.3.4 Magnetostatik

$B = \nabla \times \underline{A}$ $\quad \nabla B = 0$

$\nabla \times B = \frac{4\pi}{c} \cdot \underline{j} = \nabla(\nabla \cdot \underline{A}) - \Delta \underline{A}$

$$\mathcal{L}_{MS} = \frac{1}{8\pi}\left(\nabla \times \underline{A}\right)^2 - \frac{1}{c}\,\underline{A}\cdot\underline{j}$$

$\sum_{k\,ij\,ab} \varepsilon_{ijk}\,\varepsilon_{abk}\,(\partial_i A_j)(\partial_a A_b)$

$$= \frac{1}{8\pi}\sum_{ij}\left[(\partial_i A_j)^2 - (\partial_i A_j)(\partial_j A_i)\right] - \frac{1}{c}\sum_i A_i j_i$$

$\sum_k \varepsilon_{ijk}\,\varepsilon_{abk} = \delta_{ia}\delta_{jb} - \delta_{ib}\delta_{ja}$

$$\Rightarrow \frac{1}{4\pi}\Delta A_k - \frac{1}{4\pi}\partial_k(\nabla\cdot A) + \frac{1}{c}\cdot j_k = 0$$

8.3.5. Elektrodynamik

Dynamik $\quad \underline{E} = -\nabla\Phi - \frac{1}{c}\partial_t\underline{A}$

$$\mathcal{L} = \frac{1}{8\pi}\left[\left(\nabla\Phi + \frac{1}{c}\partial_t\underline{A}\right)^2 - (\nabla\times\underline{A})^2\right] \underbrace{- \rho\Phi + \frac{1}{c}\,\underline{A}\cdot\underline{j}}_{-\frac{1}{c} j_\mu A^\mu} \hat{=} \mathcal{L}_{ES}\Big|_{\nabla\Phi \to \nabla\Phi + \frac{1}{c}\partial_t\underline{A}} - \mathcal{L}_{MS}$$

$$j^\mu = \begin{pmatrix} c\rho \\ j_x \\ j_y \\ j_z \end{pmatrix} \qquad A^\mu = \begin{pmatrix} \Phi \\ A_x \\ A_y \\ A_z \end{pmatrix}$$

$\underline{E}^2 - \underline{B}^2$?

$$F_{\mu\nu}F^{\mu\nu} = -F_{\mu\nu}F^{\nu\mu} = -\mathrm{Tr}\left\{(F_{\mu\nu})(F^{\mu\nu})\right\}$$

$$= \dots = 2\underline{B}^2 - 2\underline{E}^2$$

$$\Rightarrow \mathcal{L} = -\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu} - \frac{1}{c} j_\mu A^\mu$$ manifest kovariant

Eichtrafo $A^\mu \to A^\mu + \partial^\mu \Lambda(\underline{r}, t)$ ← skalare Fkt $\qquad F^{\mu\nu} = (\partial^\mu A^\nu - \partial^\nu A^\mu)$ ist invariant

$$\int d^3x\,dt\left[j_\mu \partial^\mu \Lambda\right] \overset{P.I.}{\underset{4}{=}} -\int d^4x\,\Lambda\left[\underbrace{\partial^\mu j_\mu}_{0}\right] = 0$$

$\Rightarrow$ Wirkung S ist Lorentz- & Eichinvariant

$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} j^\nu$ $\qquad$ homog. aut. erfüllt

$$\mathcal{L} = -\frac{1}{8\pi}\left[(\partial_\alpha A_\beta)(\partial^\alpha A^\beta) - (\partial_\alpha A_\beta)(\partial^\beta A^\alpha)\right] - \frac{1}{c} j_\alpha A^\alpha$$

ELG: $$0 = \partial^\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial^\mu A^\nu)}\right) - \frac{\partial\mathcal{L}}{\partial A^\nu}$$ $\qquad \partial_\alpha A_\beta = \eta_{\alpha\sigma}\eta_{\beta\kappa}\partial^\sigma A^\kappa$

$$= -\frac{1}{4\pi} \partial^\mu \partial_\mu A_\nu + \frac{1}{4\pi} \partial_\nu \partial^\mu A_\mu + \frac{1}{c} j_\nu$$

$$\Rightarrow \partial^\mu F_{\mu\nu} = \frac{4\pi}{c} j_\nu$$