Wdh. • PT in EM-Feld rel.

Vierergeschw. Viererpot.

$$L(\underline{q},\dot{\underline{q}}) = \frac{1}{\gamma}\left[-m_0 c^2 - \frac{e}{c}\, u_\mu \cdot A^\mu\right]$$

$$S = \int_{t_1}^{t_2} L\, dt = \int_{\tau_1}^{\tau_2} \gamma \cdot d\tau\; L(\dot{q}, q) \quad \text{ist Lorentz-inv.}$$

• Übergang zur $\mathcal{L}$ $\qquad L = \int d^3r\, \mathcal{L}$

$$S = \int d^4x\, \mathcal{L} \qquad \mathcal{L} \text{ ist ein Lorentz-Skalar}$$

$$\Rightarrow \frac{1}{c}\partial_t \frac{\partial \mathcal{L}}{\partial(\partial_t u_i)} + \sum_{k=1}^{n-1} \partial_k \frac{\partial \mathcal{L}}{\partial(\partial_k u_i)} - \frac{\partial \mathcal{L}}{\partial u_i} = 0$$

$$\boxed{\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu u_i)} - \frac{\partial \mathcal{L}}{\partial u_i} = 0}$$

$\ell = n \cdot a$

• Beispiel 1D Kette $\quad 1 \quad \overset{\leftrightarrow}{a} \quad \ldots \quad n$

$$L = \frac{1}{2} m \sum_i \dot{q}_i^2 - \frac{1}{2} k \sum_i (q_{i+1} - q_i)^2 = \int_0^\ell \underbrace{\left[\frac{1}{2}\mu (\partial_t u)^2 - \frac{1}{2} Y (\partial_x u)^2\right]}_{\mathcal{L}} dx$$

• ES: $\mathcal{L}_{ES} = \frac{1}{8\pi}(\underbrace{\nabla \Phi}_{-E_{ES}})^2 - \rho \cdot \Phi \quad \Rightarrow \Delta \Phi = -4\pi \rho$

$\Phi \hat{=} u_i$

• MS: $\mathcal{L}_{MS} = \frac{1}{8\pi}(\nabla \times \underline{A})^2 - \frac{1}{c}\underline{A} \cdot \underline{j}$

• $A^\mu = \begin{pmatrix} \Phi \\ A_x \\ A_y \\ A_z \end{pmatrix} \qquad j^\mu = \begin{pmatrix} c\rho \\ j_x \\ j_y \\ j_z \end{pmatrix} \qquad$ rel. VZ $\qquad j_\mu A^\mu = c\rho \cdot \Phi - \underline{j} \cdot \underline{A}$

$$\Rightarrow \mathcal{L} = \frac{1}{8\pi}\left[\left(\nabla \Phi + \frac{1}{c}\partial_t \underline{A}\right)^2 - (\nabla \times \underline{A})^2\right] - \frac{1}{c}\left[c\rho \cdot \Phi - \underline{j} \cdot \underline{A}\right]$$

$$= -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} - \frac{1}{c} j_\mu A^\mu$$

allg. Konstruktionshilfe: Welche Lorentz-Skalare kann man bilden (sinnvoll)

8.3.6. Noether-Theorem für Felder

kont. Symmetrie, welche S inv. lässt

allg. $\mathcal{L} \to \mathcal{L} + \delta \mathcal{L} = \mathcal{L} + \underbrace{\partial_\mu f^\mu}_{\text{Vierer-Divergenz}}$ lässt S invariant

infinitesimal kleine Änderung der Felder

$u_i \to u_i + \delta u_i$

$$\Rightarrow \delta \mathcal{L} = \sum_i \frac{\partial \mathcal{L}}{\partial u_i}\delta u_i + \sum_i \frac{\partial \mathcal{L}}{\partial(\partial_\mu u_i)} \delta(\partial_\mu u_i)$$

$$= \sum_i \underbrace{\left[\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu u_i)}\right]}_{EL} \delta u_i + \sum_i \frac{\partial \mathcal{L}}{\partial(\partial_\mu u_i)} \partial_\mu(\delta u_i) = \sum_i \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu u_i)}\delta u_i\right] = \partial_\mu f^\mu$$

ELG $\tilde{j}^\mu = \left[\sum_i \frac{\partial \mathcal{L}}{\partial(\partial_\mu u_i)} \delta u_i - j^\mu\right]$

$$\Rightarrow \boxed{\partial_\mu \tilde{j}^\mu = 0}$$ verallg. kont.-gleichung

in d. Regel inf. klein

$$\tilde{Q} = \int d^3r \left[c \sum_i \frac{\partial \mathcal{L}}{\partial(\partial_t u_i)} \delta u_i - j^0\right]$$

Bsp $x^\mu \to x^\mu + \delta x^\mu$

$$u_i(x^\mu) \to u_i(x^\mu + \delta x^\mu) = u_i(x^\mu) + \underbrace{\partial^\nu u_i(x^\mu)\,\delta x_\nu}_{\delta u_i = (\partial^\nu u_i(x^\mu))\delta x_\nu} + \dots$$

$$\delta \mathcal{L} = (\partial_\mu \mathcal{L})\,\delta x^\mu = \partial_\mu j^\mu$$

$$j^\mu = \mathcal{L}\,\delta x^\mu = \eta^{\mu\nu} \mathcal{L}\,\delta x_\nu$$

$$\partial_\mu \underbrace{\left[\sum_i \frac{\partial \mathcal{L}}{\partial(\partial_\mu u_i)} (\partial^\nu u_i) - \eta^{\mu\nu} \mathcal{L}\right]}_{T^{\mu\nu}} \delta x_\nu = 0 \qquad \forall\, \delta x_\nu$$

$T^{\mu\nu}$ kanonischer Energie-Impuls-Tensor

$$\Rightarrow \partial_\mu T^{\mu\nu} = 0$$

- nicht eindeutig bestimmt
- für Deckungsgleichheit mit dem bek. E-I-T muss man Oberf.-Terme addieren

Bsp Forderung $\eta_{\rho\sigma} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \eta_{\mu\nu}$ $\quad (\eta^{\mu\nu}) = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

$\Rightarrow$ 6 Erhaltungsgrößen 3 × Drehimpuls
3 × Lorentz-boosts

8.3.7. kombinierter Formalismus

a.) Quellen geg. $\longrightarrow$ Felder $\hat{=}$ $\mathcal{L}$

b.) Felder geg $\longrightarrow$ Dynamik einer PL $\hat{=}$ L

c.) $\rho(r,t) = \sum_i e_i\, \delta(\underline{r} - \underline{q}_i(t))$ $\qquad \underline{j}(r,t) = \sum_i e_i\, \dot{\underline{q}}_i\, \delta(\underline{r} - \underline{q}_i(t))$

Ladung Ort

$$L = -\sum_i m_{0,i} c^2 \sqrt{1 - \frac{\dot{q}_i^2}{c^2}} + \int \mathcal{L}_{ED}\, d^3r$$

$= \dots$

8.3.8. Hamilton-Dichte der ED

$$\Pi_i = \frac{\partial \mathcal{L}}{\partial(\partial_t u_i)}$$ kanon. konj. Feldimpuls

$$\mathcal{H}(u_i, \Pi_i, \partial_k u_i, x, t) = \sum_i \Pi_i (\partial_t u_i) - \mathcal{L} \quad \text{Legendre-Trafo}$$

für ED für $\rho = 0$ $\underline{j} = 0$ Vakuum

$$\mathcal{L} = \frac{1}{8\pi}\left[\left(\nabla\Phi + \frac{1}{c}\partial_t A\right)^2 - (\nabla \times A)^2\right]$$

$$\Pi_0 = \frac{\partial \mathcal{L}}{\partial(\partial_t \Phi)} = 0 \qquad \Pi_i = \frac{\partial \mathcal{L}}{\partial(\partial_t A_i)} = \underbrace{\left(\nabla\Phi + \frac{1}{c}\partial_t A\right)_i}_{-E_i} \frac{1}{4\pi c} = -\frac{1}{4\pi c} \cdot E_i$$

$$\mathcal{H} = \underline{\Pi}(\partial_t \underline{A}) - \mathcal{L} = -\frac{1}{4\pi c} \underline{E} \cdot \dot{\underline{A}} - \frac{1}{8\pi}\left[E^2 - B^2\right] \qquad -\frac{1}{c}\dot{\underline{A}} = \underline{E} + \nabla\Phi$$

$$= \frac{1}{4\pi}(\nabla\Phi) \cdot \underline{E} + \frac{1}{8\pi}\left[\underline{E}^2 + \underline{B}^2\right] \qquad (\nabla\Phi) \cdot E = \nabla(\Phi \cdot \underline{E}) \quad \text{wegen } \nabla \cdot E = 0$$

$$H = \int \mathcal{H} \, d^3 r$$

$$= \frac{1}{4\pi}\int \nabla(\Phi \cdot \underline{E})\, d^3 r + \frac{1}{8\pi}\int (E^2 + B^2)\, d^3 r$$

$$= \underbrace{\frac{1}{4\pi} \oint \Phi (\underline{E} \cdot d\underline{F})}_{\longrightarrow 0} + \frac{1}{8\pi}\int (E^2 + B^2)\, d^3 r \qquad \int_0^\infty \frac{1}{r^{1+\varepsilon}}\, dr < \infty \quad \text{für } \varepsilon > 0$$

$$\underline{E} \sim r^{-(3/2 + \varepsilon)} \qquad \varepsilon > 0$$

$$\underline{\Phi} \sim r^{-(1/2 + \varepsilon)}$$

$$\boxed{H = \frac{1}{8\pi}\int (E^2 + B^2)\, d^3 r} \Rightarrow \mathcal{H} = \frac{1}{8\pi}\left[\underline{E}^2 + \underline{B}^2\right] \hat{=} w \Big|_{\substack{B = H \\ D = E}}$$

$$= \frac{1}{8\pi}\left[\underline{E}^2 + (\nabla \times A)^2\right]$$

8.4 Quantisierung

$$H \longrightarrow \hat{H} \text{ bzw. } A \longrightarrow \hat{A} \qquad E \longrightarrow \hat{E}$$

Korrespondenz-Prinzip keine expl. Zeitabhängigkeit von A

möglichst einfach: keine Quellen $\rho = 0$ $\underline{j} = 0$ $\Big\} \Rightarrow \underline{\Phi} = 0$

spez. Coulomb-Eichung $\nabla \cdot A = 0$ $\qquad \Box \underline{A} = 0$

$$\underline{E} = -\frac{1}{c}\partial_t \underline{A} \qquad \underline{B} = \nabla \times \underline{A} \qquad \left\{\Delta - \frac{1}{c^2}\partial_t^2\right\} A_i = 0$$

Wollen: Operator $\hat{A}$ für A, so dass gilt

$$\boxed{\hat{H} = \frac{1}{8\pi}\int \left[\frac{1}{c^2}(\partial_t \hat{A})^2 + (\nabla \times \hat{A})^2\right] d^3 r}$$

$$\frac{d\hat{A}}{dt} = i\left[\hat{H}, \hat{A}\right] \iff \Box A = 0$$

Annahme: $\underline{A}$ sei periodisch auf Volumen L^3 $\rightarrow$ endlich viele Freiheitsgrade

$$\underline{A}(x+n_x L, y+n_y L, z+n_z L, t) = \underline{A}(x,y,z,t) \qquad \text{später } L \to \infty$$

Ansatz: $$\underline{A}(\underline{r},t) = \sum_{\underline{k}} \sum_{\underline{\lambda}_{\text{Polarisation}}} \sqrt{\frac{2\pi\hbar c}{k}} \Big[\underbrace{A_\lambda(\underline{k},t)}_{\text{skalare Koeff. } \in \mathbb{C}} \frac{e^{+i\underline{k}\underline{r}}}{L^{3/2}} + A_\lambda^*(\underline{k},t) \frac{e^{-i\underline{k}\underline{r}}}{L^{3/2}} \Big] \underbrace{\underline{u}_\lambda^{\underline{k}}}_{\text{Polarisation}}$$

Coulomb-Eichung $\nabla \cdot \underline{A} = 0 \Rightarrow \underline{k} \cdot \underline{u}_\lambda^{\underline{k}} = 0 \quad \forall \lambda$

$$\underline{u}_\lambda^{\underline{k}} \cdot \underline{u}_\sigma^{\underline{k}} = \delta_{\lambda\sigma}$$

$\underline{k}$ kann nur diskrete Werte annehmen

$$\underline{k} = \frac{2\pi}{L} \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} \qquad n_i \in \mathbb{Z}$$

$$\Box A = 0$$

$$\rightarrow -\underline{k}^2 A_\lambda(\underline{k},t) = \frac{1}{c^2} \partial_t^2 A_\lambda(\underline{k},t)$$

$\Rightarrow$ löse $A_\lambda(\underline{k},t) = A_\lambda(\underline{k}) \cdot e^{-i\omega(\underline{k}) \cdot t}$ 1 harmon. Oszillator mit $\omega = |\underline{k}| \cdot c$

$$A(r,t) = \sum_k \sum_\lambda \sqrt{\frac{2\pi\hbar c^2}{\omega(\underline{k})}} \frac{1}{L^{3/2}} \Big[A_\lambda(\underline{k}) e^{+i(\underline{k}\underline{r} - \omega t)} + A_\lambda^*(\underline{k}) e^{-i(\underline{k}\underline{r} - \omega t)} \Big] \underline{u}_\lambda^{\underline{k}}$$

zu Hamilton-Funktion

$$\frac{1}{L^3} \int e^{\pm i(\underline{k}-\underline{q})\cdot \underline{r}} d^3r = \delta_{\underline{k}\underline{q}} \qquad \frac{1}{L^3} \int e^{\pm i(\underline{k}+\underline{q})\cdot \underline{r}} d^3r = \delta_{-\underline{k},\underline{q}}$$

$$\underline{u}_\lambda^{\underline{k}} \cdot \underline{u}_\sigma^{\underline{k}} = \delta_{\lambda\sigma}$$

$$\underline{u}_\lambda^{\underline{k}} \cdot \underline{u}_\sigma^{-\underline{k}} = \delta_{\lambda\sigma}$$

$$\Rightarrow \boxed{H = E = \frac{1}{2} \sum_{\underline{k}} \sum_\lambda \hbar\omega(\underline{k}) \underbrace{\Big[A_\lambda(\underline{k}) A_\lambda^*(\underline{k}) + A_\lambda^*(\underline{k}) A_\lambda(\underline{k}) \Big]}_{a a^+ + a^+ a}}$$