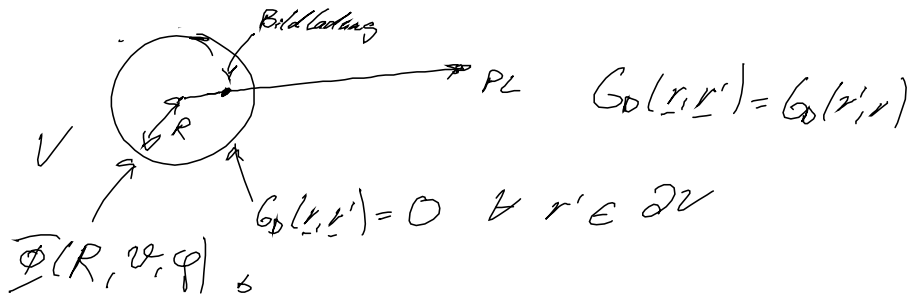


- Klausur Fr. 15.02.2019 8⁰⁰ - 10⁰⁰
Raum A 157

Wdh



- $(g|f) = \int_a^b g^*(x) f(x) dx$

orthonormal $\left[\int_a^b g_n^*(x) g_m(x) dx = \delta_{nm} \right]$
 Vollständigkeitsrel. $\left[\sum_n g_n^*(y) g_n(x) = \delta(x-y) \right]$

- Fourier-Reihen $\left. \begin{aligned} g_n^*(x) &= \frac{1}{\sqrt{2\pi}} \sin(n \cdot x) \\ g_n^*(x) &= \frac{1}{\sqrt{2\pi}} \cos(n \cdot x) \\ g_0^*(x) &= \frac{1}{\sqrt{2\pi}} \end{aligned} \right\} n \in \{1, 2, \dots\}$

alternativ $g_n(x) = \frac{1}{\sqrt{2\pi}} e^{-i n \cdot x}$

$$f(x) = \sum_n a_n g_n(x) \approx \sum_{n \in \mathbb{N}} a_n \cdot g_n(x)$$

- $\frac{1}{\sqrt{1-2x \cdot t + t^2}} = \sum_{n=0}^{\infty} P_n(x) \cdot t^n$

$$t = \frac{r'}{r} \quad x = \cos \vartheta \in [-1, +1]$$

sind Lsg einer DGL

$$(1-x^2) P_n''(x) - 2x P_n'(x) + n(n+1) P_n(x) = 0$$

• radiale DGL $r^2 R''(r) + 2r R'(r) - \ell(\ell+1) R(r) = 0$

$$R(r) = Ar^k + \frac{B}{r^{\ell+1}}$$

$$r^2 \left[k(k-1) \cdot A \cdot r^{k-2} + B(k+1)(k+2) \frac{1}{r^{k+3}} \right] + 2r \left[A \cdot k \cdot r^{k-1} - \frac{B(k+1)}{r^{k+2}} \right] - \ell(\ell+1) \left[A \cdot r^k + \frac{B}{r^{\ell+1}} \right] = 0$$

$$\underline{\ell = k(k+1)}$$

• Winkel-DGL

$$\frac{d}{d\vartheta} (\sin \vartheta P'(\vartheta)) + \ell(\ell+1) \sin \vartheta P(\vartheta) = 0$$

$$x = \cos \vartheta \quad \frac{d}{d\vartheta} = \frac{dx}{d\vartheta} \frac{d}{dx} = -\sin \vartheta \frac{d}{dx} = -\sqrt{1-x^2} \frac{d}{dx}$$

$$\frac{d}{dx} [(1-x^2) P'(x)] + \ell(\ell+1) P(x) = 0$$

"Legendre-DGL" hat $P_n(x)$ als Lösung

$$\boxed{\Phi(r, \vartheta) = \sum_{n=0}^{\infty} A_n \cdot r^n P_n(\cos \vartheta) + \sum_{n=0}^{\infty} \frac{B_n}{r^{n+1}} P_n(\cos \vartheta)}$$

allg. Lösung der Laplace-Gleichung bei axiale Symmetrie
 A_n, B_n folgen aus RB

Bsp: Kugel $\Phi(r, \vartheta)$

$$\Phi(\infty, \vartheta) = 0 \rightarrow A_n = 0$$

Radius d. Kugel

$$B_n = \int_{-1}^1 \frac{2}{z} R^{n+1} P_n(x) \Phi(R, x) dx$$

↑
bekannt

2.3. Kugelflächenfunktionen

Lösung der vollen 3D-Laplace-Gleichung

$$\Phi(r, \vartheta, \varphi) = R(r) \cdot Y(\vartheta, \varphi)$$

$$0 = \underbrace{\frac{1}{R(r)} \frac{\partial}{\partial r} \left(r^2 \frac{\partial R}{\partial r} \right)}_{+\ell(\ell+1)} + \underbrace{\frac{1}{Y(\vartheta, \varphi)} \left[\frac{1}{\sin \vartheta} \frac{\partial}{\partial \vartheta} \left(\sin \vartheta \frac{\partial Y}{\partial \vartheta} \right) + \frac{1}{\sin^2 \vartheta} \frac{\partial^2 Y}{\partial \varphi^2} \right]}_{=-\ell(\ell+1)}$$

Winkel PDE

$$\left[\frac{1}{\sin \vartheta} \frac{\partial}{\partial \vartheta} \left(\sin \vartheta \frac{\partial}{\partial \vartheta} \right) + \frac{1}{\sin^2 \vartheta} \frac{\partial^2}{\partial \varphi^2} \right] \gamma = -l(l+1) \cdot \gamma(\vartheta, \varphi)$$

heute Ansatz: $\gamma(\vartheta, \varphi) = \Theta(\vartheta) \Phi(\varphi)$

→ 2 ordn. DGL, heute Sep.-konstante

$$\frac{\partial^2 \Phi(\varphi)}{\partial \varphi^2} = -m^2 \Phi(\varphi) \rightarrow \Phi(\varphi) = \frac{e^{i \cdot m \cdot \varphi}}{\sqrt{2\pi}}$$

$$\Phi(\varphi + 2\pi) \stackrel{!}{=} \Phi(\varphi) \Rightarrow m \in \mathbb{Z}$$

Azimut-Gleichung

$$\left[\frac{\sin \vartheta}{\Theta(\vartheta)} \frac{\partial}{\partial \vartheta} \left(\sin \vartheta \frac{\partial \Theta}{\partial \vartheta} \right) + l(l+1) \cdot \sin^2 \vartheta \right] \Theta = -m^2$$

$$x = \cos \vartheta \quad \frac{d}{d\vartheta} = -\sqrt{1-x^2} \frac{d}{dx} \quad \sin \vartheta = \sqrt{1-x^2}$$

$$(1-x^2) \Theta''(x) - 2x \Theta'(x) + \left[l(l+1) - \frac{m^2}{1-x^2} \right] \Theta(x) = 0$$

für $l=0 \rightarrow$ Legendre-DGL

$$\Theta(x) = (1-x^2)^{m/2} \cdot U_m(x)$$

↑
substant

$$(1-x^2) U_m'(x) - 2(l+1) \cdot x \cdot U_m'(x) + [l(l+1) - m(m+1)] U_m = 0$$

$$(1-x^2) \cdot U_m''(x) - 2(l+2) \cdot x \cdot U_m'(x) + [l(l+1) - (m+1)(m+1)] U_m'(x) = 0$$

$$U_m'(x) = U_{m+1}(x) \rightarrow U_m(x) = \frac{d^m}{dx^m} U_0(x)$$

$$\Rightarrow \Theta(x) = (1-x^2)^{m/2} \frac{d^m}{dx^m} P_l(x) \quad \begin{matrix} l \in \{0, 1, \dots\} \\ m \in \{0, 1, \dots, l\} \end{matrix}$$

Konvention

$$P_{l,m}(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_l(x)$$

assoziierte Legendre-Polynome

$$= \frac{(-1)^m}{2^l \cdot l!} (1-x^2)^{m/2} \frac{d^{l+m}}{dx^{l+m}} (x^2-1)^l$$

assoz. Rodriguez-Formel

$$l \in \{-l, -l+1, \dots, +l-1, l\}$$

Def $Y_{\ell m}(\vartheta, \varphi) = \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}} \underbrace{P_{\ell}^m(\cos\vartheta)}_{\propto \Theta(\vartheta)} \underbrace{e^{im\varphi}}_{\propto \Phi(\varphi)}$

Kugel (Flächenfunktion)

$$\int_0^{\pi} d\vartheta \int_0^{2\pi} \sin\vartheta d\varphi Y_{\ell m}^*(\vartheta, \varphi) Y_{\ell' m'}(\vartheta, \varphi) = \delta_{\ell\ell'} \delta_{mm'}$$

$$\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y_{\ell m}^*(\vartheta, \varphi) Y_{\ell m}(\vartheta, \varphi) = \delta(\cos\vartheta - \cos\vartheta') \delta(\varphi - \varphi')$$

$$\left[\frac{1}{\sin\vartheta} \frac{\partial}{\partial\vartheta} \left(\sin\vartheta \frac{\partial}{\partial\vartheta} \right) + \frac{1}{\sin^2\vartheta} \frac{\partial^2}{\partial\varphi^2} \right] Y_{\ell m}(\vartheta, \varphi) = -\ell(\ell+1) Y_{\ell m}(\vartheta, \varphi) = -\ell^2 Y_{\ell m}$$

die ersten

$$Y_{00} = \frac{1}{\sqrt{4\pi}}$$

$$Y_{1,-1} = +\sqrt{\frac{3}{8\pi}} \sin\vartheta e^{-i\varphi} \quad Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\vartheta \quad Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin\vartheta e^{+i\varphi}$$

$$Y_{\ell, k=0}(\vartheta, \varphi) = \sqrt{\frac{2\ell+1}{4\pi}} \cdot P_{\ell}(\cos\vartheta)$$

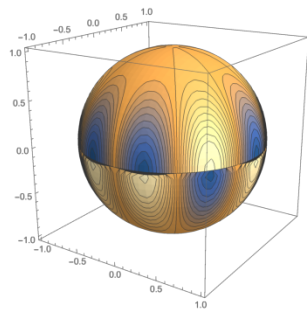
$$Y_{\ell, -k}(\vartheta, \varphi) = (-1)^k Y_{\ell, k}^*(\vartheta, \varphi)$$

$$g(\vartheta, \varphi) = \sum_{\ell=0}^{\infty} \sum_{k=-\ell}^{\ell} a_{\ell k} Y_{\ell k}(\vartheta, \varphi)$$

$$a_{\ell k} = \int_0^{\pi} d\vartheta \int_0^{2\pi} \sin\vartheta d\varphi Y_{\ell k}^*(\vartheta, \varphi) g(\vartheta, \varphi)$$

ES: $\Delta \Phi(r, \vartheta, \varphi) = 0$

$$\Phi(r, \vartheta, \varphi) = \sum_{\ell=0}^{\infty} \sum_{k=-\ell}^{\ell} \left[a_{\ell k} r^{\ell} + \frac{b_{\ell k}}{r^{\ell+1}} \right] Y_{\ell k}(\vartheta, \varphi)$$



Re $Y_{55}(\vartheta, \varphi)$

2.4. Additionstheoreme

$$\frac{1}{|\underline{r}-\underline{r}'|} \quad \underline{r} = r \begin{pmatrix} \sin \vartheta \cos \varphi \\ \sin \vartheta \sin \varphi \\ \cos \vartheta \end{pmatrix} \quad \underline{r}' = r' \begin{pmatrix} \sin \vartheta' \cos \varphi' \\ \sin \vartheta' \sin \varphi' \\ \cos \vartheta' \end{pmatrix}$$

$$\frac{1}{|\underline{r}-\underline{r}'|} = \sum_{\ell=0}^{\infty} \sum_{k=-\ell}^{+\ell} \sum_{k'=-\ell}^{+\ell} A_{\ell k k'}(r, r') \underline{Y}_{\ell k}^*(\vartheta, \varphi) \underline{Y}_{\ell k'}(\vartheta', \varphi')$$

$$= \sum_{\ell=0}^{\infty} \frac{1}{r_>} \left(\frac{r_<}{r_>} \right)^{\ell} P_{\ell}(\cos \gamma) \quad \text{korrekter: } \gamma = \angle(\underline{r}, \underline{r}')$$

$$\Delta \frac{1}{|\underline{r}-\underline{r}'|} = -4\pi \delta(\underline{r}-\underline{r}') = -\frac{4\pi}{r^2} \delta(r-r') \delta(\cos \vartheta - \cos \vartheta') \delta(\varphi - \varphi')$$

$$= -\frac{4\pi}{r^2} \delta(r-r') \sum_{\ell=0}^{\infty} \sum_{k=-\ell}^{+\ell} \underline{Y}_{\ell k}^*(\vartheta', \varphi') \underline{Y}_{\ell k}(\vartheta, \varphi)$$

$$\left(\frac{1}{r} \frac{d^2}{dr^2} r - \frac{\ell(\ell+1)}{r^2} \right) A_{\ell k k'}(r, r') \underline{Y}_{\ell k}^*(\vartheta', \varphi') \underline{Y}_{\ell k}(\vartheta, \varphi)$$

$$\Rightarrow A_{\ell k k'}(r, r') = A_{\ell k}(r, r') \cdot \delta_{\ell \ell'} \delta_{k k'}$$

$$\Rightarrow \left(\frac{1}{r} \frac{d^2}{dr^2} r - \frac{\ell(\ell+1)}{r^2} \right) A_{\ell k}(r, r') = -\frac{4\pi}{r^2} \delta(r-r')$$

ist lösbar