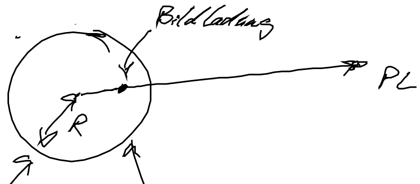
- Klausur Fr. 15.02.2019 $8^{00} - 10^{00}$
  Raum A 157

- Wdh



V

$$G_D(\underline{r}, \underline{r}') = G_D(\underline{r}', \underline{r})$$

$$G_D(\underline{r}, \underline{r}') = 0 \quad \forall\, r' \in \partial V$$

$$\underline{\Phi(R, \vartheta, \varphi)}$$

- $$(g|f) = \int_a^b g^*(x)\, f(x)\, dx$$

orthonormal $\int_a^b g_m^*(x)\, g_n(x)\, dx = \delta_{nm}$

Vollständigkeitsrel. $\sum_n g_n^*(y)\, g_n(x) = \delta(x-y)$

- Fourier-Reihen
$$g_n^s(x) = \frac{1}{\sqrt{\pi}} \sin(n \cdot x)$$
$$g_n^c(x) = \frac{1}{\sqrt{\pi}} \cos(n \cdot x)$$
$$\Bigg\}\ n \in \{1, 2, \dots\}$$
$$g_0^c(x) = \frac{1}{\sqrt{2\pi}}$$

<u>alternativ</u> $g_n(x) = \frac{1}{\sqrt{2\pi}} e^{i n \cdot x}$

$$f(x) = \sum_n a_n\, g_n(x) \approx \sum_{n<N} a_n \cdot g_n(x)$$

- $$\frac{1}{\sqrt{1 - 2x \cdot t + t^2}} = \sum_{n=0}^{\infty} P_n(x) \cdot t^n$$

$$t = \frac{r'}{r} \qquad x = \cos\vartheta \in [-1, +1]$$

sind Lsg einer DGL

$$(1-x^2)\, P_n''(x) - 2x\, P_n'(x) + n(n+1)\, P_n(x) = 0$$

$$\int_{-1}^{+1} P_n(x) P_m(x)\,dx = N_n \cdot \delta_{nm}$$

$$\int_{-1}^{+1} \frac{dx}{1-2\cdot t x + t^2} = \sum_{n,m=0}^{\infty} \int dx\, P_n(x) P_m(x) \cdot t^n t^m$$

$$= \sum_{n=0}^{\infty} t^{2n} \underbrace{\int_{-1}^{+1} |P_n(x)|^2 dx}_{N_n}$$

$$y = 1-2tx+t^2 \qquad = \frac{1}{2t}\int_{(1-t)^2}^{(1+t)^2} \frac{dy}{y} = \frac{1}{t}\ln\left(\frac{1+t}{1-t}\right) = \underline{2\sum_{n=0}^{\infty} \frac{t^{2n}}{2n+1}}$$

$y(x=-1) = (1+t)^2$

$y(x=+1) = (1-t)^2$

$$\int_{-1}^{+1} |P_n(x)|^2 dx = \frac{2}{2n+1}$$

$$\boxed{\int_{-1}^{+1} P_n(x)\, P_m(x)\, dx = \frac{2\,\delta_{nm}}{2n+1}}$$

$$f(x) = \sum_n a_n P_n(x)$$

$$a_n = \frac{2n+1}{2}\int_{-1}^{+1} P_n(x) \cdot f(x)\, dx$$

2.2. Lösung der Laplace-Gleichung bei axialer Symmetrie

$$\text{KK}: \Delta f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta \frac{\partial f}{\partial \vartheta}\right) + \underbrace{\frac{1}{r^2\sin^2\vartheta}\frac{\partial^2 f}{\partial\varphi^2}}_{=0}$$

$$f = f(r, \vartheta)$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial\Phi}{\partial\vartheta}\right) = 0$$

$$\Phi(r,\vartheta) = R(r)\circ P(\vartheta)$$

$$\frac{P(\vartheta)}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{R(r)}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial P}{\partial\vartheta}\right) = 0$$

$$\underbrace{\left[\frac{1}{R(r)}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)\right]}_{+\ell} + \underbrace{\left[\frac{1}{P(\vartheta)\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial P}{\partial\vartheta}\right)\right]}_{-\ell} = 0$$

- radiale DGL $r^2 R''(r) + 2r R'(r) - \xi \cdot R(r) = 0$

$$R(r) = A \cdot r^n + \frac{B}{r^{n+1}}$$

$$r^2 \left[n(n-1) \cdot A \cdot r^{n-2} + B(n+1)(n+2) \frac{1}{r^{n+3}} \right]$$
$$+ 2r \left[A \cdot n \cdot r^{n-1} - \frac{B(n+1)}{r^{n+2}} \right] - \xi \left[A \cdot r^n + \frac{B}{r^{n+1}} \right] = 0$$

$$\underline{\xi = n(n+1)}$$

- Winkel-DGL

$$\frac{d}{d\vartheta} \left(\sin\vartheta \, P'(\vartheta) \right) + n(n+1) \sin\vartheta \, P(\vartheta) = 0$$

$$x = \cos\vartheta \qquad \frac{d}{d\vartheta} = \frac{dx}{d\vartheta} \frac{d}{dx} = -\sin\vartheta \frac{d}{dx} = -\sqrt{1-x^2} \frac{d}{dx}$$

$$\frac{d}{dx} \left[(1-x^2) P'(x) \right] + n(n+1) P(x) = 0$$

"Legendre-DGL" hat $P_n(x)$ als Lösung

$$\boxed{\Phi(r,\vartheta) = \sum_{n=0}^{\infty} A_n \cdot r^n P_n(\cos\vartheta) + \sum_{n=0}^{\infty} \frac{B_n}{r^{n+1}} \cdot P_n(\cos\vartheta)}$$

allg. Lösung der Laplace-Gleichung bei axialer Symmetrie

A_n B_n folgen aus RB

Bsp.: Kugel $\Phi(R,\vartheta)$ $\Phi(\infty,\vartheta) = 0 \rightarrow A_n = 0$

Radius d. Kugel

$$B_n = \int_{-1}^{+1} \frac{2n+1}{2} R^{n+1} P_n(x) \, \Phi(R,x) \, dx$$

bekannt

2.3. Kugelflächenfunktionen

Lösung der vollen 3d-Laplacegleichung

$$\Phi(r,\vartheta,\varphi) = R(r) \cdot Y(\vartheta,\varphi)$$

$$0 = \underbrace{\frac{1}{R(r)} \frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r} \right)}_{+\xi = n(n+1)} + \underbrace{\frac{1}{Y(\vartheta,\varphi)} \left[\frac{1}{\sin\vartheta} \frac{\partial}{\partial\vartheta}\left(\sin\vartheta \frac{\partial}{\partial\vartheta} \right) + \frac{1}{\sin^2\vartheta} \frac{\partial^2}{\partial\varphi^2} \right] Y(\vartheta,\varphi)}_{= -n(n+1)}$$

Winkel PDE

$$\left[\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial}{\partial\vartheta}\right)+\frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}\right]G=-n(n+1)\cdot G(\vartheta,\varphi)$$

neuer Ansatz: $G(\vartheta,\varphi)=\Theta(\vartheta)\Phi(\varphi)$

$\to$ 2 ordinäre DGL, neue Sep.-konstante

$$\frac{\partial^2\Phi(\varphi)}{\partial\varphi^2}=-m^2\Phi(\varphi)\rightsquigarrow \boxed{\Phi(\varphi)=\frac{e^{im\varphi}}{\sqrt{2\pi}}}$$

$$\Phi(\varphi+2\pi)\stackrel{!}{=}\Phi(\varphi)\Rightarrow \boxed{m\in\mathbb{Z}}$$

Azimut-Gleichung

$$\left[\frac{\sin\vartheta}{\Theta(\vartheta)}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial\Theta}{\partial\vartheta}\right)+n(n+1)\cdot\sin^2\vartheta\right]=+m^2$$

$x=\cos\vartheta \quad \frac{d}{d\vartheta}=-\sqrt{1-x^2}\frac{d}{dx} \qquad \sin\vartheta=\sqrt{1-x^2}$

$$(1-x^2)\Theta''(x)-2x\Theta'(x)+\left[n(n+1)-\underline{\frac{m^2}{1-x^2}}\right]\Theta(x)=0$$

für $m=0\to$ Legendre-DGL

$$\Theta(x)=(1-x^2)^{m/2}\cdot \underbrace{v_m(x)}_{\text{unbekannt}}$$

$$(1-x^2)v_m''(x)-2(m+1)\cdot x\cdot v_m'(x)+[n(n+1)-m(m+1)]v_m(x)=0$$

$$(1-x^2)\cdot v_m'''(x)-2(m+2)\cdot x\cdot v_m''(x)+[n(n+1)-(m+1)(m+1)]v_m'(x)=0$$

$$v_m'(x)=v_{m+1}(x)\to v_m(x)=\frac{d^m}{dx^m}v_0(x)$$

$$\Rightarrow \Theta(x)=(1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x) \qquad n\in\{0,1,\dots\}$$

$m\in\{0,1,\dots n\}$

Konvention

$$\boxed{P_\ell^m(x)=(-1)^m(1-x^2)^{m/2}\frac{d^m}{dx^m}P_\ell(x)}$$

assoziierte Legendre-Polynome

$$=\frac{(-1)^m}{2^\ell\cdot\ell!}(1-x^2)^{m/2}\frac{d^{\ell+m}}{dx^{\ell+m}}(x^2-1)^\ell \qquad m\in\{-\ell,-\ell+1,\dots,+\ell-1,\ell\}$$

assoz. Rodrigues-Formel

Def $$Y_{lm}(\vartheta,\varphi) = \sqrt{\frac{2l+1}{4\pi}\,\frac{(l-m)!}{(l+m)!}}\;\underbrace{P_l^m(\cos\vartheta)}_{\propto\Theta(\vartheta)}\;\underbrace{e^{im\varphi}}_{\propto\Phi(\varphi)}$$

Kugelflächenfunktionen

$$\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\vartheta\, d\vartheta\; Y^*_{lm}(\vartheta,\varphi)\, Y_{l'm'}(\vartheta,\varphi) = \delta_{ll'}\cdot\delta_{mm'}$$ orthonormal

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{+l} Y^*_{lm}(\vartheta',\varphi')\, Y_{lm}(\vartheta,\varphi) = \delta(\cos\vartheta-\cos\vartheta')\cdot\delta(\varphi-\varphi')$$ Vollst

$$\left[\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial}{\partial\vartheta}\right)+\frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}\right]Y_{lm}(\vartheta,\varphi) = -l(l+1)\,Y_{lm}(\vartheta,\varphi) = r^2\Delta Y_{lm}$$

die ersten

$$Y_{00} = \frac{1}{\sqrt{4\pi}}$$

$$Y_{1,-1} = +\sqrt{\frac{3}{8\pi}}\sin\vartheta\, e^{-i\varphi} \qquad Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\vartheta \qquad Y_{11} = -\sqrt{\frac{3}{8\pi}}\sin\vartheta\, e^{+i\varphi}$$

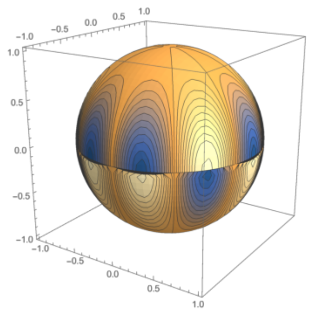
$$Y_{l,m=0}(\vartheta,\varphi) = \sqrt{\frac{2l+1}{4\pi}}\cdot P_l(\cos\vartheta)$$

$$Y_{l,-m}(\vartheta,\varphi) = (-1)^m\cdot Y^*_{l,+m}(\vartheta,\varphi)$$

$$\boxed{\begin{aligned} g(\vartheta,\varphi) &= \sum_{l=0}^{\infty}\sum_{m=-l}^{+l} a_{lm}\, Y_{lm}(\vartheta,\varphi)\\ a_{lm} &= \int_0^{2\pi} d\varphi \int_0^{\pi}\sin\vartheta\, d\vartheta\; Y^*_{lm}(\vartheta,\varphi)\, g(\vartheta,\varphi)\end{aligned}}$$

ES: $\Delta\Phi(r,\vartheta,\varphi) = 0$

$$\Phi(r,\vartheta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\left[a_{lm}\cdot r^{+l} + \frac{b_{lm}}{r^{l+1}}\right] Y_{lm}(\vartheta,\varphi)$$



Re $Y_{55}(\vartheta,\varphi)$

2.4. Additionstheorem

$$\frac{1}{|\underline{r}-\underline{r}'|} \qquad \underline{r} = r\begin{pmatrix} \sin\vartheta\cos\varphi \\ \sin\vartheta\sin\varphi \\ \cos\vartheta \end{pmatrix} \qquad \underline{r}' = r'\begin{pmatrix} \sin\vartheta'\cos\varphi' \\ \sin\vartheta'\sin\varphi' \\ \cos\vartheta' \end{pmatrix}$$

$$\frac{1}{|r-r'|} = \sum_{\ell,\ell'=0}^{\infty}\sum_{m=-\ell}^{+\ell}\sum_{m'=-\ell'}^{+\ell'} A_{\ell\ell'mm'}(r,r')\, Y^*_{\ell'm'}(\vartheta',\varphi')\, Y_{\ell m}(\vartheta,\varphi)$$

$$= \sum_{\ell=0}^{\infty}\frac{1}{r_>}\left(\frac{r_<}{r_>}\right)^{\ell} P_\ell(\cos\gamma)$$

Korrektur : $\gamma = \sphericalangle(\underline{r},\underline{r}')$

$$\Delta\frac{1}{|r-r'|} = -4\pi\,\delta(\underline{r}-\underline{r}') = -\frac{4\pi}{r^2}\,\delta(r-r')\underbrace{\delta(\cos\vartheta-\cos\vartheta')\,\delta(\varphi-\varphi')}$$

$$= -\frac{4\pi}{r^2}\,\delta(r-r')\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\vartheta',\varphi')\,Y_{\ell m}(\vartheta,\varphi)$$

$$= \sum_{\ell\ell'}\sum_{mm'}\left(\frac{1}{r}\frac{d^2}{dr^2}r - \frac{\ell(\ell+1)}{r^2}\right) A_{\ell\ell'mm'}(r,r')\,Y^*_{\ell'm'}(\vartheta',\varphi)\,Y_{\ell m}(\vartheta,\varphi)$$

$$\Rightarrow A_{\ell\ell'mm'}(r,r') = A_{\ell m}(r,r')\cdot\delta_{\ell\ell'}\,\delta_{mm'}$$

$$\Rightarrow \left(\frac{1}{r}\frac{d^2}{dr^2}r - \frac{\ell(\ell+1)}{r^2}\right)A_{\ell m}(r,r') = -\frac{4\pi}{r^2}\,\delta(r-r')$$

ist lösbar