

Tat: Fr. 16⁰⁰ - 18⁰⁰ \implies Do 16⁰⁰ - 18⁰⁰

Wdh $\cdot \Delta \Phi = 0$ bei ax. Symmetrie

$$\Phi(r, \vartheta) = R(r) \Theta(\vartheta)$$

$$= \sum_{\ell=0}^{\infty} \left[A_{\ell} \cdot r^{\ell} + \frac{B_{\ell}}{r^{\ell+1}} \right] P_{\ell}(\cos \vartheta)$$

A_{ℓ} & B_{ℓ} folgen aus RB

$\cdot \Delta \Phi = 0$ ohne ax. Sym.

$$\Phi(r, \vartheta, \varphi) = R(r) Y(\vartheta, \varphi)$$

$$r^2 \Delta Y_{\ell m}(\vartheta, \varphi) = -\ell(\ell+1) Y_{\ell m}(\vartheta, \varphi)$$

$$\Delta_{\vartheta} \Delta_{\varphi} = \int_0^{2\pi} d\varphi \int_0^{\pi} \sin \vartheta d\vartheta [\dots]$$

$$\Phi(r, \vartheta, \varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} \left[a_{\ell m} \cdot r^{\ell} + \frac{b_{\ell m}}{r^{\ell+1}} \right] Y_{\ell m}(\vartheta, \varphi)$$

$$[\dots] = Y_{\ell m}^*(\vartheta, \varphi) Y_{\ell m}(\vartheta, \varphi)$$

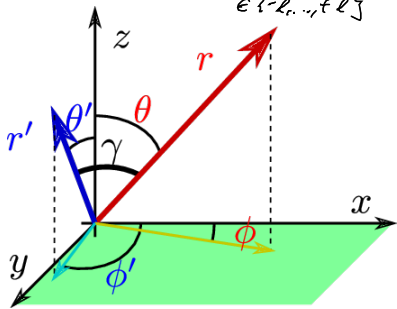
$$\frac{1}{|r-r'|} = \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} \left(\frac{r'}{r} \right)^{\ell} P_{\ell}(\cos \gamma) \quad \gamma = \angle(r, r')$$

$$\sqrt{r^2 + r'^2 - 2rr' \cos \gamma}$$

$$\underline{r} = r \begin{pmatrix} \sin \vartheta \cos \varphi \\ \sin \vartheta \sin \varphi \\ \cos \vartheta \end{pmatrix} \quad (\vartheta, \varphi, \vartheta', \varphi')$$

$$\frac{1}{|r-r'|} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} \frac{A_{\ell m}(r, r')}{r^{\ell+1}} \cdot Y_{\ell m}(\vartheta, \varphi) Y_{\ell m}^*(\vartheta', \varphi')$$

\downarrow
 $\ell' = \ell, m' = m$
 $\ell' = \ell, m' = -m$



$$\cos \gamma = \sin \vartheta \sin \vartheta' [\cos \varphi \cos \varphi' + \sin \varphi \sin \varphi'] + \cos \vartheta \cos \vartheta'$$

$$A_{\ell m}(r, r') = A_{\ell m}(r, r') \cdot \Delta_{\ell m} \Delta_{\ell m}$$

$$\left(\frac{1}{r} \frac{d^2}{dr^2} r - \frac{\ell(\ell+1)}{r^2} \right) A_{\ell}(r, r') = -\frac{4\pi}{r^2} \delta(r-r')$$

$$A_{\ell}(r, r') = \begin{cases} a_{\ell} \cdot r^{\ell} & : r < r' \\ b_{\ell} \cdot \frac{1}{r^{\ell+1}} & : r > r' \end{cases}$$

$$A_\ell(r'-\epsilon, r') = A_\ell(r'+\epsilon, r') \quad \text{Stetigkeit}$$

$$i.) \quad a_\ell \cdot r'^\ell = b_\ell \cdot \frac{1}{r'^{\ell+1}} \quad \rightarrow \quad b_\ell = a_\ell \cdot (r')^{2\ell+1}$$

$$ii.) \quad \frac{d^2}{dr^2} r A_\ell(r, r') - \frac{\ell(\ell+1)}{r} A_\ell(r, r') = -\frac{4\pi}{r} \delta(r-r') \quad \left| \int_{r'-\epsilon}^{r'+\epsilon} \dots dr \right.$$

$$\left[\frac{d}{dr} r A_\ell(r, r') \right]_{r=r'-\epsilon}^{r=r'+\epsilon} + [0] = -\frac{4\pi}{r} \quad \left. \begin{array}{l} |r=r'+\epsilon \\ |r=r'-\epsilon \\ \lim \epsilon \rightarrow 0 \end{array} \right\}$$

$$a_\ell = \frac{4\pi}{2\ell+1} \frac{1}{(r')^{2\ell+1}}$$

$$A_{\ell, \text{phys}}(r, r') = \frac{4\pi}{2\ell+1} \delta_{\ell, \ell'} \delta_{m, m'} \begin{cases} \frac{r^\ell}{r'^{\ell+1}} & : r < r' \\ \frac{r^\ell}{r'^{\ell+1}} & : r > r' \end{cases}$$

$$= \frac{4\pi}{2\ell+1} \delta_{\ell, \ell'} \delta_{m, m'} \cdot \frac{1}{r'} \left(\frac{r}{r'} \right)^\ell$$

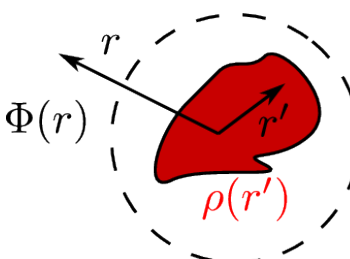
$$\frac{1}{|r-r'|} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} \frac{4\pi}{2\ell+1} \frac{1}{r'} \left(\frac{r}{r'} \right)^\ell Y_{\ell m}^*(\vartheta', \varphi') Y_{\ell m}(\vartheta, \varphi)$$

$$P_\ell(\cos \gamma) = \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{+\ell} Y_{\ell m}^*(\vartheta', \varphi') Y_{\ell m}(\vartheta, \varphi)$$

$$\cos \gamma = \cos \vartheta \cos \vartheta' + \sin \vartheta \sin \vartheta' \cdot \cos(\varphi - \varphi')$$

Add.-Theorem der KFF

2.5. Multipolentwicklung



$$\Phi(r) = \int \frac{\rho(r')}{|r-r'|} d^3 r'$$

$\rho(r > r') = 0$
entwickeln für $|r| \gg |r'|$

2.5.1. sphärische Ents.

$$\Phi(r, \vartheta, \varphi) = \int \rho(r') \frac{1}{r} \sum_{\ell=0}^{\infty} \left(\frac{r'}{r} \right)^\ell \cdot P_\ell(\cos \gamma) d^3 r'$$

$$= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} \frac{4\pi}{2\ell+1} \frac{1}{r^{\ell+1}} Y_{\ell m}(\vartheta, \varphi) \left[\int Y_{\ell m}^*(\vartheta', \varphi') \rho(r') (r')^\ell d^3 r' \right]$$

$$Q_{\ell m} = \int d^3 r' \int_0^{2\pi} \int_0^\pi \sin \vartheta' d\vartheta' \rho(r') (r')^\ell Y_{\ell m}^*(\vartheta', \varphi')$$

Multipolentwicklung

speziell $l=0$ "Monopolmoment" (1)

$l=1$, $m \in \{-1, 0, +1\}$ "Dipolmoment" (3)

$l=2$, $m \in \{-2, \dots, +2\}$ "Quadrupol-..." (5)

a.) $q_0 = \frac{1}{4\pi\epsilon_0} \int \rho(\underline{r}') d^3r' = \frac{q}{4\pi\epsilon_0}$ Monopolmoment $\hat{=}$ Gesamtladung

b.) Dipol $q_{1-1} = \int q_{1-1}^*(\vartheta', \varphi') \rho(\underline{r}') |\underline{r}'| d^3r'$
 $= \sqrt{\frac{3}{4\pi}} \int \sin\vartheta' [\cos\varphi' + i \sin\varphi'] \cdot \rho(\underline{r}') \cdot r' d^3r'$
 $= \sqrt{\frac{3}{4\pi}} \int (x' + iy') \rho(\underline{r}') d^3r'$

$q_{1+1} = \sqrt{\frac{3}{4\pi}} \int (-x' + iy') \rho(\underline{r}') d^3r'$

$q_{10} = \sqrt{\frac{3}{4\pi}} \int z' \rho(\underline{r}') d^3r'$

$\Phi(r, \vartheta, \varphi) = \frac{q}{r} + \frac{P \cdot \underline{r}}{r^3} + \dots \mathcal{O}\left(\frac{1}{r^3}\right)$

\uparrow
dipolart. Feld

$P = \int \underline{r}' \rho(\underline{r}') d^3r'$

2.5.2. Kartesische Multipol-EL

$\Phi(\underline{r}) = \int \frac{\rho(\underline{r}')}{|\underline{r} - \underline{r}'|} d^3r' = \int \rho(\underline{r}') \left[\frac{1}{|\underline{r}|} + \sum_{i=1}^3 \frac{x_i x'_i}{r^3} + \frac{1}{2!} \sum_{i,j=1}^3 \frac{3x_i x_j - \delta_{ij} r^2}{r^5} x'_i x'_j + \dots \int d^3r' \right]$

$= \frac{q}{r} + \frac{\underline{r} \cdot \underline{P}}{r^3} + \frac{1}{2!} \sum_{i,j=1}^3 \frac{3x_i x_j - \delta_{ij} r^2}{r^5} \int \rho(\underline{r}') x'_i x'_j d^3r' + \dots$

$\underline{P} = \int \underline{r}' \rho(\underline{r}') d^3r' = \int \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \rho(\underline{r}') d^3r'$ 3×3 mit 6 Komponenten

letzter Term

$$\frac{1}{3!} \sum_{ij} \frac{3x_i x_j - r^2 \delta_{ij}}{r^5} \left[\int \rho(r') \left[3x_i x_j - r^2 \delta_{ij} + r^2 \delta_{ij} \right] d^3 r' \right]$$

hatte $\sum_{ij} (3x_i x_j - r^2 \delta_{ij}) \cdot \delta_{ij} = \sum_{i=1}^3 (3x_i^2 - r^2) = 3r^2 - 3r^2 = 0$

$$\Rightarrow = \frac{1}{3!} \sum_{ij} \frac{3x_i x_j - r^2 \delta_{ij}}{r^5} Q_{ij}$$

vert. Quadrupolmoment $Q_{ij} = \int \rho(r') [3x_i x_j - r^2 \delta_{ij}] d^3 r'$

$\text{Tr}\{Q_{ij}\} = \sum_{i=1}^3 Q_{ii} = 0$ $\int \rho Q_{ii} = Q_{ii}$ 6 unabh. + Spur \rightarrow "5 übrig"

$$P = \begin{pmatrix} P_x \\ P_y \\ P_z \end{pmatrix} = \sqrt{\frac{4\pi}{3}} \begin{pmatrix} \frac{1}{\sqrt{2}} (Q_{11} - Q_{22}) \\ Q_{12} \\ \frac{1}{\sqrt{2}} (Q_{11} + Q_{22}) \\ Q_{33} \end{pmatrix}$$

Monopol

Dipol

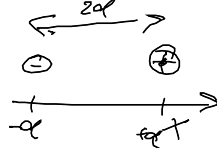
keine Multipole

höhere Terme

Quadrupol

Oktupol

Dipol $\rho = q (\delta(x-d) - \delta(x+d)) \delta(y) \delta(z)$



$$P = \int \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \rho(x', y', z') d^3 r' = \begin{pmatrix} 2 \cdot q \cdot d \\ 0 \\ 0 \end{pmatrix} \quad \text{Dipol} \hat{=} \text{Ladung mal Abstand}$$

z.B. WW-Energie im äußeren Feld

vgl. 7.8.

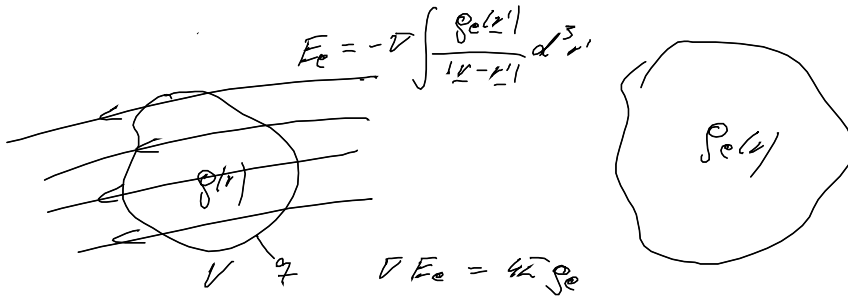
$$W = \sum_i q_i \cdot \overline{\Phi_e}(r_i) \rightarrow \int_V \rho(r) \cdot \overline{\Phi_e}(r) d^3r$$

Anfangswert
Endwert in $r=0$

E_e wird von $r=0$ weit entfernt
 $\nabla \cdot E_e \rightarrow \forall r \in V$ gilt: $\nabla \cdot E_e = 0$

$$\begin{aligned} \overline{\Phi_e} &= \overline{\Phi_e}(0) + r \cdot (\nabla \overline{\Phi_e})|_{r=0} + \frac{1}{2} \sum_{ij=1}^3 \frac{\partial^2 \overline{\Phi_e}}{\partial x_i \partial x_j} \cdot x_i x_j + \dots \\ &= \overline{\Phi_e}(0) - r \cdot (E_e(0)) - \frac{1}{2} \sum_{ij=1}^3 \frac{\partial E_{eij}}{\partial x_i} \cdot x_i x_j + \dots \end{aligned}$$

$$0 = \frac{1}{6} \cdot r^2 (\nabla \cdot E_e)$$



$$\forall r \in V: \nabla \cdot E_e = 4\pi \rho_e|_V = 0$$

$$\overline{\Phi}(r) = \overline{\Phi_e}(0) - r \cdot E_e(0) - \frac{1}{6} \sum_{ij=1}^3 (3x_i x_j - r^2 \delta_{ij}) \frac{\partial E_{eij}}{\partial x_i} \Big|_0$$

in W einsetzen

$$W = q \cdot \overline{\Phi_e}(0) - p \cdot E_e(0) - \frac{1}{6} \sum_{ij=1}^3 Q_{ij} \frac{\partial E_{eij}}{\partial x_i} \Big|_0 + \dots$$

Monopol mit Potential

Dipol mit Feldstärke

Quadrupol mit Hh. der Feldst.

Beispiel Pot. eines ^{1000q} Dipols als Ausgangspunkt

$$\overline{\Phi}_1(r) = \frac{P_1 \cdot r}{r^3} = \frac{P_1^x \cdot x + P_1^y \cdot y + P_1^z \cdot z}{[x^2 + y^2 + z^2]^{3/2}}$$

$$E_1(r) = -\nabla \overline{\Phi}_1(r) = - \left[\frac{P_1}{r^3} - \frac{3(P_1 \cdot r) \cdot r}{r^5} \right]$$

$$\hat{z} = \frac{r}{|r|}$$

$$= - \frac{P_1 - 3(\hat{z} \cdot P_1) \cdot \hat{z}}{r^3} \stackrel{!}{=} E_{ext}$$

WW-Energie zwischen 2 Dipolen

$$W_{12} = -P_2 \cdot E_1 = + \frac{P_2 \cdot P_1 - 3(\hat{z} \cdot P_2)(\hat{z} \cdot P_1)}{r^3}$$

