English Summary:

classical Van der Pol osc.: $\ddot{Q} + \omega_0^2 Q - \varepsilon(1-2Q^2)\dot{Q} = 0 \qquad \varepsilon > 0$

neg. lin. damping   pos. nonlin. damping

network of VdP osc. (nonlocal coupling of range R):

$\dot{x}_k = F(x_k) + \frac{\sigma}{2R}\sum_{j=k-R}^{k+R} H(x_j - x_k)$, $k = 1,\dots,N$, $x_k = \begin{pmatrix} Q \\ P \end{pmatrix} \in \mathbb{R}^2$, $P = \dot{Q}$

H 2x2 coupling matrix

$\Rightarrow$ chimera states

$\varepsilon \ll 1$: $\dot{\alpha} = \frac{\varepsilon}{2}(1 - |\alpha|^2)\alpha$ Stuart-Landau osc. $\alpha \equiv \frac{1}{\sqrt{2}}(Q + iP)$

Quantum Van der Pol osc.:

$\dot{\rho}_s = 2\kappa_1\left(a^+ \rho_s a - \frac{1}{2}\{\rho_s, a a^+\}\right) + 2\kappa_2\left(a^2 \rho_s (a^+)^2 - \frac{1}{2}\{\rho_s, (a^+)^2 a^2\}\right)$

1-photon absorption   2-photon emission

4.7.2 Quantum Van der Pol oscillator

Bastidas, Omelchenko, Zakharova, Schöll, Brandes, PRE 92, 062924 (28.12.2015)

— " — , in "Control of Self-Organizing Nonlinear Systems" (eds. Schöll, Klapp, Hövel) Springer 2016

Aim: Lindblad Master eq. with dissipation rates κ_1 (linear neg. damping) and κ_2 (nonlinear loss)

Decomposition of bosonic operators

$$\boxed{a(t) = \underbrace{\tilde{a}}_{\substack{\text{quantum}\\ \text{fluctuations}\\ \text{annih. op. } \tilde{a}}} + \underbrace{\alpha(t)}_{\substack{\text{mean field}\\ \alpha = \langle\alpha| a |\alpha\rangle \in \mathbb{C}\\ \text{semiclass. trajectory}}}} \qquad a^+(t) = \tilde{a}^+ + \alpha^*(t)$$

coherent state α

$a|\alpha\rangle = \alpha|\alpha\rangle$

Quantum fluctuations about a semiclassical trajectory $\alpha(t)$:

Density matrix in the co-moving frame (co-rotating with $\alpha(t)$):

$$\rho_\alpha(t) = D^+(\alpha)\,\rho(t)\,D(\alpha) \qquad \rho \equiv \rho_s$$

with displacement op. (unitary) $D(\alpha) = e^{\alpha a^+ - \alpha^* a} = e^{\alpha \tilde{a}^+ - \alpha^* \tilde{a}}$

$$= e^{-\frac{|\alpha|^2}{2}}\, e^{\alpha\tilde{a}^+}\, e^{-\alpha^*\tilde{a}}$$

$$\dot{\rho}_\alpha(t) = -\frac{i}{\hbar}\left[H^{(\alpha)}(t), \rho_\alpha(t)\right] + 2\kappa_1 D^+(\alpha)\left(\tilde{a}^+\rho\,\tilde{a} - \frac{1}{2}\{\rho, \tilde{a}\tilde{a}^+\}\right)D(\alpha)$$

$$+ 2\kappa_2 D^+(\alpha)\left(\tilde{a}^2\rho\,(\tilde{a}^+)^2 - \frac{1}{2}\{\rho, (\tilde{a}^+)^2\tilde{a}^2\}\right)D(\alpha)$$

$$= -\frac{i}{\hbar}[H^{(\alpha)}(t), \varrho_\alpha(t)] + \hat{\mathcal{L}}_1\varrho_\alpha + \hat{\mathcal{L}}_2\varrho_\alpha$$

Liouville operators of dissipation

with $H^{(\alpha)}(t) := -i\hbar D^+(\alpha)\partial_t D(\alpha)$

$$= -\frac{i\hbar}{2}[\dot\alpha(t)\alpha^*(t) - \alpha(t)\dot\alpha^*(t)]$$

$$-i\hbar[\dot\alpha(t)\tilde a^+ - \dot\alpha^*(t)\tilde a]$$

<u>gaussian</u> quantum fluctuations for $|\alpha(t)| \gg 1$: semiclassical high-density limit

$\Rightarrow$ neglect all non-Gaussian (non-quadratic) fluctuations

$$\hat{\mathcal{L}}_1\varrho_\alpha = 2\kappa_1\left(\tilde a^+\varrho_\alpha\tilde a - \frac{1}{2}\{\varrho_\alpha, \tilde a\tilde a^+\}\right) - \frac{i}{\hbar}\cancel{[i\hbar\kappa_1\alpha\tilde a^+, \varrho_\alpha]} \quad \text{lin. in } \tilde a^+ \text{ or } \tilde a$$

incoherent (dissip.) part

$$-\frac{i}{\hbar}\cancel{[-i\hbar\kappa_1\alpha^*\tilde a, \varrho_\alpha]}$$

coherent part

$\hat{\mathcal{L}}_2\varrho_\alpha$ analogously

$$\Rightarrow \boxed{\begin{aligned}\dot\varrho_\alpha(t) &= -i\kappa_2[i(\alpha^*)^2\tilde a^2 - i\alpha^2(\tilde a^+)^2, \varrho_\alpha] \\ &+ 2\kappa_1\left(\tilde a^+\varrho_\alpha\tilde a - \frac{1}{2}\{\varrho_\alpha, \tilde a\tilde a^+\}\right) + 8\kappa_2|\alpha|^2\left(\tilde a\varrho_\alpha\tilde a^+ - \frac{1}{2}\{\varrho_\alpha, \tilde a^+\tilde a\}\right)\end{aligned}} \quad (1)$$

quantum fluctuations

position of co-moving frame:

$$(2) \quad \boxed{\dot\alpha(t) = \kappa_1\alpha(t) - 2\kappa_2\alpha(t)|\alpha(t)|^2} \quad \text{semiclassical trajectory}$$

<u>Interpretation:</u>

initially coherent state $\varrho(t=0) = |\alpha(0)\rangle\langle\alpha(0)|$

$\Leftrightarrow \varrho_\alpha(t=0) = |0\rangle\langle 0|$ vacuum state in co-moving frame centered at $\alpha(0)$

$\hat{=}$ initial condition for Eq. (1)

Sol. of semiclassical Eq. (2) $\hat{=}$ classical Stuart-Landau eq.

$$\dot\alpha = \frac{\varepsilon}{2}(1 - |\alpha|^2)\alpha$$

with $\kappa_1 = \frac{\varepsilon}{2}$, $\kappa_2 = \frac{\varepsilon}{4}$

$\Rightarrow$ (i) <u>limit cycle</u> $\boxed{|\alpha|^2 = \frac{\kappa_1}{2\kappa_2}}$

(ii) time-dependent <u>squeezing</u> described by master eq. (1)

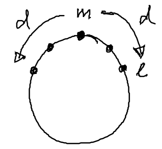
4.7.3 <u>Network of quantum Van der Pol osc.</u>

Ring network of N Van der Pol osc. a_ℓ, a_ℓ^+ , $\ell = 1, \dots, N$
indices mod N

$$(3) \quad \boxed{\dot{\varrho} = -\frac{i}{\hbar}[H, \varrho] + 2\sum_{\ell=1}^{N} \left[\kappa_1 \mathcal{D}(a_\ell^+) + \kappa_2 \mathcal{D}(a_\ell^2)\right]}$$

with dissipative processes $\mathcal{D}(\hat{O}) := \hat{O}\varrho\hat{O}^+ - \frac{1}{2}\{\varrho, \hat{O}^+\hat{O}\}$ "dissipator"

<u>nonlocal</u> coupling of range d:
new!



$$H = \hbar \sum_{\substack{m=1 \\ m \neq \ell}}^{N} K_{\ell m}\left(a_\ell^+ a_m + a_\ell a_m^+\right)$$

coupling matrix $K_{\ell m} = \frac{V}{2d}\,\Theta(d - |\ell - m|)$

coupling strength V

general form of Lindblad master eq.

$$\dot{\varrho} = -\frac{i}{\hbar}[H, \varrho] + \sum_\mu \gamma_\mu \left(L_\mu \varrho L_\mu^+ - \frac{1}{2}\{\varrho, L_\mu^+ L_\mu\}\right)$$

Lindblad op. $L_\mu = a_\mu^+$ (lin.)
$L_\mu = a_\mu^2$ (nonlin.)

eff. Hamiltonian

$$H_{eff} := H - \frac{i\hbar}{2}\sum_\mu \gamma_\mu L_\mu^+ L_\mu$$

$$(3) \Rightarrow H_{eff} = i\hbar\kappa_1 \sum_{\ell=1}^{N}\left(a_\ell^+ a_\ell + 1\right) - i\hbar\kappa_2 \sum_{\ell=1}^{N} \hat{n}_\ell(\hat{n}_\ell - 1) \qquad \hat{n}_\ell := a_\ell^+ a_\ell$$

$$+ \hbar \sum_{\substack{m=1 \\ m \neq \ell}}^{N} K_{\ell m}\left(a_\ell^+ a_m + a_\ell a_m^+\right)$$

(nonlocal coupling)

Bose–Hubbard model with long-range interaction
and complex on-site self-energies and chem. pot.
($\Rightarrow$ application to driven dissipative Bose-Einstein condensation)

<u>Gaussian quantum fluctuations</u>

expansion $b_\ell(t) := D^+(\alpha(t))\, a_\ell\, D(\alpha(t)) = \underbrace{\tilde{a}_\ell}_{\text{quantum fluct.}} + \underbrace{\alpha_\ell(t)}_{\text{mean-field}}$

with displacement op. $D(\alpha) = e^{\alpha\tilde{a}^+ - \alpha^*\tilde{a}}$

$$\alpha(t) \equiv (\alpha_1, \alpha_2, \dots, \alpha_N)$$

$$\tilde{a} \equiv (\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_N)$$

Semiclassical regime $\quad |\alpha_\ell(t)| \gg \|\tilde{a}_\ell\|$ quantum fluct. gaussian

expansion as Eq. (1) $\Rightarrow$ Master eq. in co-moving frame, describes quantum fluctuations

$$\Rightarrow (4) \quad \boxed{\dot{\rho}_\alpha \approx \underbrace{-\frac{i}{\hbar}[H_Q^{(\alpha)}, \rho_\alpha]}_{\text{coherent dyn.}} + 2\sum_{\ell=1}^{N}[\kappa_1 \mathcal{D}(\tilde{a}_\ell^+) + 4\kappa_2 |\alpha_\ell|^2 \mathcal{D}(\tilde{a}_\ell)]}$$

$$H_Q^{(\alpha)}(t) = \hbar \sum_{\ell=1}^{N} \Big(i\kappa_2 (\alpha_\ell^*)^2 \tilde{a}_\ell^2 - i\kappa_2 \alpha_\ell^2 (\tilde{a}_\ell^+)^2 + \sum_{r=1}^{N} K_{\ell\,\ell+r} (\tilde{a}_\ell^+ \tilde{a}_{\ell+r} + \tilde{a}_\ell \tilde{a}_{\ell+r}^+) \Big)$$

$$+ \hbar \sum_{\ell=1}^{N} \left(-i[\dot{\alpha}_\ell \tilde{a}_\ell^+ - \dot{\alpha}_\ell^* \tilde{a}_\ell] + i\kappa_1 \alpha_\ell \tilde{a}_\ell^+ - i\kappa_1 \alpha_\ell^* \tilde{a}_\ell \right)$$

$$+ \hbar \sum_{\ell=1}^{N} \left(2i\kappa_2 \alpha_\ell (\alpha_\ell^*)^2 \tilde{a}_\ell - 2i\kappa_2 \alpha_\ell^* \alpha_\ell^2 \tilde{a}_\ell^+ \right)$$

$$+ \hbar \sum_{r=1}^{N} K_{\ell,\ell+r} \left(\alpha_{\ell+r} \tilde{a}_\ell^+ + \alpha_{\ell+r}^* \tilde{a}_\ell + \alpha_\ell^* a_{\ell+r} + \alpha_\ell \tilde{a}_{\ell+r}^+ \right)$$

semiclass. trajectory ($\Rightarrow$ lin. terms in Master eq. vanish):

$$(5) \quad \boxed{\dot{\alpha}_\ell(t) = \alpha_\ell(t)(\kappa_1 - 2\kappa_2 |\alpha_\ell(t)|^2) - i \sum_{\substack{s=1 \\ s \neq \ell}}^{N} K_{\ell s} \alpha_s(t)}$$

$$(4) \Rightarrow \frac{d}{dt}\langle \tilde{a}_i \rangle \equiv \mathrm{tr}\left[\tilde{a}_i \dot{\rho}_\alpha(t)\right] = \kappa_1 \langle \tilde{a}_i \rangle - 4\kappa_2 |\alpha_i|^2 \langle \tilde{a}_i \rangle - 2\kappa_2 \alpha_i^2 \langle \tilde{a}_i^+ \rangle - i \sum_{\substack{s=1 \\ s \neq i}}^{N} K_{is} \langle \tilde{a}_s \rangle$$