

English Summary:

Ornstein-Uhlenbeck process

$$\dot{x} = -kx + \sqrt{2D} \xi(t) \quad \text{correl. fun. } G(\tau) \equiv \langle x(t)x(t+\tau) \rangle = \frac{D}{k} e^{-k|\tau|}$$

$$\text{power spectral density } S(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\tau e^{-i\omega\tau} G(\tau) = \frac{D}{\pi} \frac{1}{\omega^2 + k^2}$$

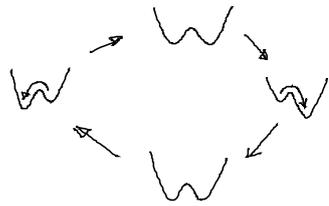
$$\text{direct sol. of Langevin eq.: } x(t) = x(0)e^{-kt} + \sqrt{2D} \int_0^t e^{-k(t-t')} \xi(t') dt'$$

$$\langle x(t)x(t') \rangle \approx \frac{D}{k} e^{-k|t-t'|} \quad (t, t' \rightarrow \infty)$$

3. Noise-induced oscillations and patterns

3.1 Stochastic resonance

$$\dot{x} = -V'(x) + A_0 \cos(\Omega t) + \sqrt{2D} \xi(t) \Rightarrow \langle x(t) \rangle = \bar{x} \cos(\Omega t - \bar{\phi})$$

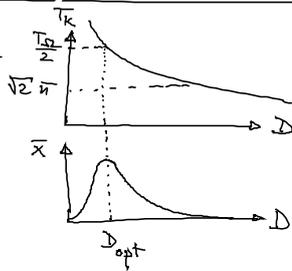


$$\Omega = \pi \nu_k(D)$$

$$\text{Kramers' rate } \nu_k(D) = \frac{1}{T_k} \quad \text{mean first passage time}$$

$$\frac{2\pi}{\Omega} = 2T_k(D)$$

Stochast. Resonanz



für kleine Amplituden:

$$\bar{x}(D) = \frac{A_0 \langle x^2 \rangle_0}{D} \frac{2\nu_k(D)}{\sqrt{4\nu_k^2(D) + \Omega^2}}$$

$$\bar{\phi}(D) = \arctan \frac{\Omega}{2\nu_k} \quad \text{phase lag}$$

Spektrale Leistungsdichte

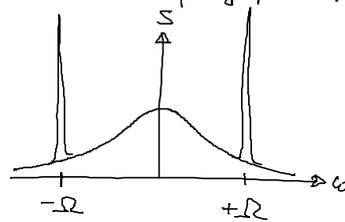
$$S(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\tau e^{-i\omega\tau} \langle x(t+\tau)x(t) \rangle \quad \leftarrow \text{zusätzlich über Anfangsphase gemittelt}$$

$$\text{Untergrundrauschen } S_N(\omega) \approx \frac{4\nu_k \langle x^2 \rangle_0}{4\nu_k^2 + \omega^2}$$

überlagert durch δ -Peaks

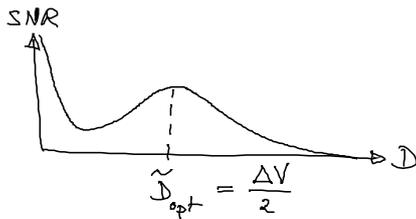
bei $\omega = \pm \Omega$

$$S(\omega) = \frac{\pi}{2} \bar{x}(D)^2 [\delta(\omega - \Omega) + \delta(\omega + \Omega)] + S_N(\omega)$$



Signal-to-noise ratio (Maß für Signalverstärkung):

$$\text{SNR} = \frac{2 \lim_{\Delta\omega \rightarrow 0} \int_{\Omega-\Delta\omega}^{\Omega+\Delta\omega} S(\omega) d\omega}{S_N(\Omega)} \approx \pi \left(\frac{A_0 x_m}{D} \right)^2 \nu_k(D) \sim \frac{e^{-\frac{\Delta V}{D}}}{D^2}$$



$$\tilde{D}_{opt} \neq D_{opt}$$

3.2 Rauschinduzierte Oszillationen

Jetzt: autonome Systeme, ohne externe periodische treibende Kraft

Annahme: determinist. System hat stabilen Fixpt.

⇒ Rauschen kann Oszillationen induzieren

(self-sustained oscillations, stoch. limit cycle)
(LC)

Reviews: Lindner, Garcia-Ojalvo, Neirame, Solimanoky-Geyer:

Effects of noise in excitable systems, Phys. Rep. 392, 321 (2004)

Janson, Balazso, Schöll: Control of noise-induced dynamics.

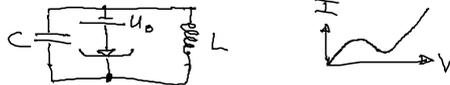
In: Handbook of Chaos Control (Wiley, 2008)

häufig unterhalb einer Bif. eines determinist. Grenzzyklus

(Hopf-Bifurk., globale Bifurkation (SNIPER = saddle-node infinite period)
= SNIC)

1. Beispiel: Van der Pol-Osz. (1920: nichtlin. el. Stromkreis)

- System knapp unterhalb einer Hopf-Bifurkation



$$\begin{cases} \dot{x} = y \\ \dot{y} = (\varepsilon - x^2)y - \omega_0^2 x + \sqrt{2D} \tilde{\zeta}(t) \end{cases}$$

$$\Leftrightarrow \ddot{x} - \underbrace{(\varepsilon - x^2)}_{\text{nichtlin. Reib.}} \dot{x} + \omega_0^2 x = \tilde{D} \tilde{\zeta}(t)$$

NB: Rauschintens. $\tilde{D} = \sqrt{2D}$

$D=0$ (determ.): Fixp. $x^* = y^* = 0$

Stab. des Fixp. (linearisiert) $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$, $A = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & \varepsilon \end{pmatrix}$

$$\lambda^2 - \lambda \text{tr} A + \det A = 0, \text{tr} A = \varepsilon, \det A = \omega_0^2 > 0$$

$$\Rightarrow \left. \begin{array}{l} \varepsilon = 0 \text{ Hopf-Bif. } (\lambda = \pm i\omega_0) \\ \varepsilon < 0 \text{ stabiler Fokus} \\ \varepsilon > 0 \text{ instab. Fokus + LC} \end{array} \right\} \lambda = \frac{\varepsilon}{2} \pm i\sqrt{\omega_0^2 - \left(\frac{\varepsilon}{2}\right)^2}$$

Wähle z.B. $\varepsilon = -0.01$, $\omega_0 = 1$

⇒ rauschinduz. Osz. ($D \neq 0$)



Fig. 1 - Simulated phase portraits of noise-induced oscillations of the Van der Pol system for $\varepsilon = -0.01$, $\omega_0 = 1$, $\tau = 0$, $K = 0$: a) $\tilde{D} = 0.003$; b) $\tilde{D} = 0.5$. Integration time: 300 time units.

Pomplun et al.
Europhys. Lett.
71, 366 (2005)

2. Beispiel: FitzHugh-Nagumo-Modell
(Bomhoeffer-van-der-Pol)

- anregbares System (Typ II): Schwellenverhalten (stabiler Fixp.)

Anwendung: Spiking von Neuronen

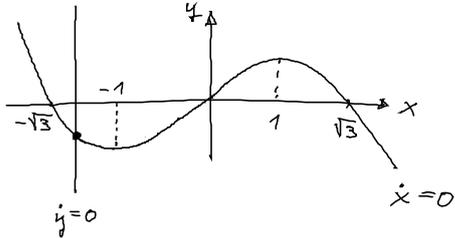
$$\begin{cases} \varepsilon \dot{x} = x - \frac{x^3}{3} - y \\ \dot{y} = x - y \end{cases} \quad \text{Aktivator (schnell)}$$

$$\boxed{y = x + a + D\tilde{f}(t)}$$

Inhibitor (langsam)
 Zeitskalenverh. $\epsilon \ll 1$
 Anreg. Schwelle a

$D=0$: Fixp. $x = -a, y = -a + \frac{a^3}{3}$

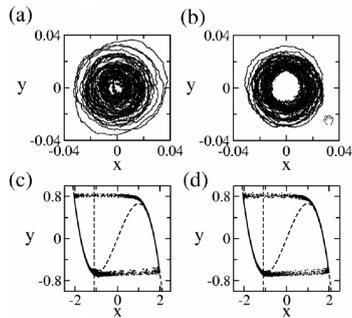
stab. $\begin{pmatrix} \delta \dot{x} \\ \delta \dot{y} \end{pmatrix} = \frac{1}{\epsilon} \begin{pmatrix} 1-a^2 & -1 \\ \epsilon & 0 \end{pmatrix}, \quad \text{tr} A = 1-a^2, \quad \det A = \epsilon > 0$



$a = 1$: Hopf-Bif.
 $a < 1$: instab. Fixp. + LC (oszillator. Regime)
 $a > 1$: stab. Knoten (anregbares Regime)
 (z.B. $a = 1.1, \epsilon = 0.01$)
 slow-fast-System

$D \neq 0$: rauschinduz. Ose.

Janson, Balanov, Schöll: PRL 93, 010601 (2004)
 Balanov, Janson, Schöll: Physica D 179, 1 (2004)
 Schöll, Balanov, Janson, Verma: Stoch. Dyn. 5, 281 (2005)



Janson (2004)

Van der Pol

FitzHugh-Nagumo

FIG. 1. Phase portraits of noise-induced motion: (a),(b) Van der Pol oscillator at $D = 0.003$; (c),(d) FitzHugh-Nagumo system at $D = 0.09$ (the dashed lines denote the null isoclines). (a),(c) $K = 0$; (b),(d) $K = 0.2, \tau = T_0$.

3.3. Kohärenzresonanz

gang, Ditinger, Ning, Haken: Stoch. resonance without external periodic forcing, PRL 71, 807 (1993)

Pikovsky, Kurths: Coherence resonance in a noise-driven excitable system, PRL 78, 775 (1997)

Verma, Sapsis, Stone: PRE 56, 270 (1997)

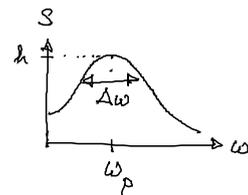
- konstruktiver Einfluss von Rauschen
- Regularität („Kohärenz“) der rauschinduz. Ose. am größten für endl. Rauschintensität D_{opt}

• Maß für Regularität:

- Signal-Rausch-Verhältnis (SNR) (Haken)

$$\beta = \frac{h}{\Delta\omega/\omega_p}$$

quality factor



- Standardabweich. der ISI T_{ISI}
 (interspike interval)



$$R_T = \frac{\sqrt{\langle T_{|s|}^2 \rangle - \langle T_{|s|} \rangle^2}}{\langle T_{|s|} \rangle}$$

normierte $|S|$ -Fluktuation

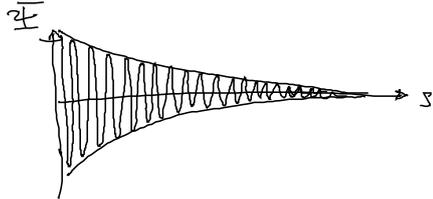
- Korrelationszeit $t_{\text{cor}} := \frac{1}{\overline{\Phi(0)}} \int_0^{\infty} |\overline{\Phi(s)}| ds$

(Autokorrel. fkt. $\overline{\Phi(s)} := \langle [x(t+s) - \langle x \rangle][x(t) - \langle x \rangle] \rangle$
 Varianz $\overline{\Phi(0)} = \sigma^2$)

Motivation der Def.:

für lin. stoch. Prozesse $\dot{x} = -(k + i\omega_0)x + \xi(t)$

$\overline{\Phi(s)} = \overline{\Phi(0)} e^{-ks} \cos(\omega_0 s)$ (S. 2.3. stat. Ornstein-Uhl.,
 $\text{Re } e^{-(k+i\omega_0)s}$)



exp. abklingende Enveloppe
 Trägerfrequ. ω_0

$k > 0$ (stabiler Fixp.)

Zus. hang zwischen t_{cor} und k :

$$t_{\text{cor}} = \int_0^{\infty} e^{-ks} |\cos \omega_0 s| ds$$

Approx. $\frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \cos \phi d\phi = \frac{2}{\pi}$ für $k \ll \omega_0$ (Füllfaktor)

$$\Rightarrow t_{\text{cor}} \approx \frac{2}{\pi} \int_0^{\infty} e^{-ks} ds = \frac{2}{\pi k}$$

also $\overline{\Phi(s)} = \overline{\Phi(0)} e^{-\frac{2}{\pi} \frac{s}{t_{\text{cor}}}} \cos(\omega_0 s)$

$k = |\text{Re}(\text{Eigenwert des Fixp.})| = \text{Bifurk. par.}$
 $\text{Re } \lambda < 0$ (Abstand vom Hopf-Bif.)

\Rightarrow je stabiler der Fixp., umso kürzer die Korrel.zeit
 (je weiter weg von der Hopf-Bif.)