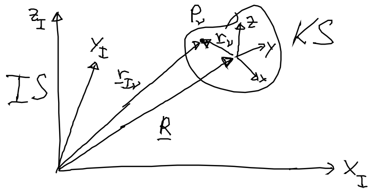
10.1 Kinematik

a) Koord. system



Eulerscher Satz: $\dot{\underline{r}}_{I\nu}(t) = \dot{\underline{R}}(t) + \underline{\omega}(t) \times \underline{r}_\nu(t) \quad \forall P_\nu \quad (10.4)$

unabh. von Wahl von $\underline{R}$

10.2. Dynamik des starren Körpers (I) & Trägheitstensor

a) Dynam. Grundgleichungen:

- innere Kräfte: $\underline{F}^{(i)} = \sum_{\nu\mu} \underline{F}_{\nu\mu} = 0 \quad (8.6)$

$$\underline{D}^{(i)} = \sum_{\nu\mu} \underline{r}_{I\nu} \times \underline{F}_{\nu\mu} = 0 \quad (8.7)$$

(1) Gesamtmasse: $M = \sum_\nu m_\nu \quad (10.10)$

(2) " impuls: $\underline{P} = \sum_\nu m_\nu \dot{\underline{r}}_{I\nu} \quad (10.11)$

(3) " drehimpuls: $\underline{L}_I = \sum_\nu m_\nu \underline{r}_{I\nu} \times \dot{\underline{r}}_{I\nu} \quad (10.12)$

- Bewegungsgln.:

$$\dot{\underline{P}} = \sum_\nu \underline{F}_\nu^{(a)} = \underline{F}^{(a)} \quad (10.13) \; [(8.8)]$$

$$\dot{\underline{L}}_I = \sum_\nu \underline{r}_{I\nu} \times \underline{F}_\nu^{(a)} = \sum_\nu \underline{D}_\nu^{(a)} = \underline{D}^{(a)} \quad (10.14) \; [(8.14)]$$

... 6 Dgl., die die Dynamik des starren Körpers ($f = 6$) eindeutig bestimmen

- Gleichgewichtsbed.: $\left.\begin{array}{l}\underline{F}^{(a)} = 0 \\ \underline{D}^{(a)} = 0\end{array}\right\} \left\{\begin{array}{l}\dot{\underline{P}} = 0 \\ \dot{\underline{L}}_I = 0\end{array}\right. \longrightarrow$ Grundlage der Statik

- im folgenden: 2 Fälle:

$\boxed{\begin{array}{l}\text{(i) Anf.pkt. } \underline{R} = \text{Schwerpkt. } \underline{R}_s = \frac{\sum_\nu m_\nu \underline{r}_{I\nu}}{M} \quad (10.15) \\ \quad \text{Bsp: Erde, frei fallender Kreisel} \\ \text{(ii) } \underline{R} \text{ mit } \dot{\underline{R}} = 0 \\ \quad \text{Bsp: Kreisel mit festem Auflagepkt.}\end{array}}$

Folgerungen mit (10.4): $\dot{\underline{r}}_{I\nu}(t) = \dot{\underline{R}}(t) + \underline{\omega}(t) \times \underline{r}_\nu(t)$

b) Impuls:

(i) $\underline{R} = \underline{R}_s$: $\quad \underline{P} = \sum_\nu m_\nu \dot{\underline{r}}_{I\nu} \overset{(10.4)}{=} M \dot{\underline{R}}_s + \underline{\omega} \times \underbrace{\sum_\nu m_\nu \underline{r}_\nu}_{=0,\ \text{wegen}\ \underline{R}=\underline{R}_s}$

$$\longrightarrow \boxed{\underline{P} = M \dot{\underline{R}}_s} \quad (10.16)$$

(ii) $\dot{\underline{R}} = 0$ $\qquad \boxed{\underline{P} = \underline{\omega} \times \sum_\nu m_\nu \underline{r}_\nu = \underline{\omega} \times M \underline{r}_s} \quad (10.17)$

c) Drehimpuls

$$\underline{L}_I = \sum_\nu m_\nu \underline{r}_{I\nu} \times \dot{\underline{r}}_{I\nu} \qquad (*)$$

$\underline{r}_{I\nu} = \underline{R} + \underline{r}_\nu$, $\quad \dot{\underline{r}}_{I\nu} = \dot{\underline{R}} + \underline{\omega} \times \underline{r}_\nu$

(i) $\underline{R} = \underline{R}_s$: $\xrightarrow[\sum_\nu m_\nu \underline{r}_\nu = 0]{}$

$$\boxed{\begin{aligned} \underline{L}_I &= \underline{L}_s + \underline{L} && (10.18)\\ \text{mit } \underline{L}_s &= M \underline{R}_s \times \dot{\underline{R}}_s \overset{(10.16)}{=} \underline{R}_s \times \underline{P} && (10.19)\\ \underline{L} &= \sum_\nu m_\nu \underline{r}_\nu \times (\underline{\omega} \times \underline{r}_\nu) && (10.20) \end{aligned}}$$

$\underline{L}_I$ ... Drehimpuls bzgl. Ursprung von IS
$=\underline{L}_s$ ... " des Schwerpkts " "
$+\underline{L}$ ... " des starren Körpers bzgl. Aufpkt $\equiv \underline{R}_s$

(ii) $\dot{\underline{R}} = 0$: $\xrightarrow{(*)}$ $\quad \underline{L}_I = \underbrace{\underline{R} \times (\underline{\omega} \times \underbrace{\sum_\nu m_\nu \underline{r}_\nu}_{M \underline{r}_s})}_{\underline{P}\ [\text{s. } (10.17)]} + \underbrace{\underline{L}}_{(10.20)}$

$$\boxed{\underline{L}_I = \underline{L}_s + \underline{L} = \underline{R} \times \underline{P} + \underline{L}} \quad \text{wie } (10.18) - (10.20)$$

mit $\underline{L}$ bzgl $\underline{R}$ (10.21)

d) Trägheitstensor: Umschreibung von (10.20)

[Einschub: MMP (SS19): Tensoren 2. Stufe
Kapitel 4: 9.5/16.5.19]

- Hilfsformel: $\underline{a} \times (\underline{b} \times \underline{a}) = |\underline{a}|^2 \underline{b} - \underline{a}(\underline{a} \cdot \underline{b})$
$\quad = |\underline{a}|^2 \underline{b} - (\underline{a} \otimes \underline{a})\, \underline{b}$
$\quad = (|\underline{a}|^2 \underline{1} - \underline{a} \otimes \underline{a})\, \underline{b} \quad (10.22)$

$[\underline{a} \times (\underline{b} \times \underline{c}) = \underline{b}(\underline{a} \cdot \underline{c}) - \underline{c}(\underline{a} \cdot \underline{b})]$

[NB: $\underline{u} \otimes \underline{v}$ ... dyadisches Produkt von $\underline{u}, \underline{v}$; spezieller Tensor 2. Stufe

$(\underline{u} \otimes \underline{v})\, \underline{b} = \underline{u}\ \underline{v} \cdot \underline{b} \quad (10.23)$]

- Drehimpuls $\underline{L}$: (10.20) mit (10.22) ($\underline{\omega} = \underline{b}$, $\underline{a} = \underline{r}_\nu$)

$$\boxed{\underline{L} = \underline{\underline{\Theta}}\, \underline{\omega} \quad \text{mit } \underline{\underline{\Theta}}(t) = \sum_\nu m_\nu \left[|\underline{r}_\nu|^2 \underline{1} - \underline{r}_\nu(t) \otimes \underline{r}_\nu(t) \right]} \quad (10.24)$$

... Trägheitstensor in koordin. unabh. Form

(i) $\underline{\underline{\Theta}}$ ... Eigenschaft des starren Körpers

(ii) lineare Abb: $\underline{L} \nearrow \underline{\omega}$

- Komponentendarstellung bzgl. ONB $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$

$$\underline{e}_i \cdot \Big| \; \underline{L} = \underline{\underline{\Theta}} \; \underbrace{\underline{\omega}}_{= \omega_j \underline{e}_j} \longrightarrow L_i = \underbrace{\underline{e}_i \cdot \underline{\underline{\Theta}} \, \underline{e}_j}_{\Theta_{ij}} \omega_j$$

$$\longrightarrow \boxed{L_i = \Theta_{ij} \, \omega_j \quad \text{mit } \Theta_{ij} = \underline{e}_i \cdot \underline{\underline{\Theta}} \, \underline{e}_j = \sum_\nu m_\nu \left[|\underline{r}_\nu|^2 \delta_{ij} - x_{\nu i} x_{\nu j} \right]} \quad (10.25)$$

[Bsp. s. Übung]

$\longrightarrow \underline{\underline{\Theta}} = \Theta_{ij} \, \underline{e}_i \otimes \underline{e}_j$!

Matrix $\underline{\underline{\Theta}}$ mit $[\underline{\underline{\Theta}}]_{ij} = \Theta_{ij}$!!

- symm. Tensor 2. St.: $\Theta_{ij} = \Theta_{ji} \longrightarrow$ 6 unabh. Komponenten

Hauptdiagonale: $\Theta_{11}, \Theta_{22}, \Theta_{33}$ ... Trägheitsmomente

Nichtdiagonale: $\Theta_{12}, \Theta_{23}, \Theta_{13}$ ... Deviationsmomente

$\longrightarrow$ Lagerkräfte Bsp. Rad

- Trägheitsmoment bzgl. Achse $\hat{\underline{\nu}}$ mit $|\hat{\underline{\nu}}| = 1$:

$$\Theta_{\nu\nu} = \hat{\underline{\nu}} \cdot \underline{\underline{\Theta}} \, \hat{\underline{\nu}} \quad (10.26)$$

- $\underline{\underline{\Theta}} = \underline{\underline{\Theta}}(t)$: Wo steckt Zeitabh. relativ zu IS

(i) ONB des KS: $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$ ... körperfest!

$$\underline{\underline{\Theta}}(t) = \Theta_{ij} \, \underline{e}_i(t) \otimes \underline{e}_j(t), \quad \Theta_{ij} = \underbrace{\underline{e}_i(t) \cdot \underline{\underline{\Theta}}(t) \, \underline{e}_j(t)}_{\text{zeitunab.}} \quad (10.27)$$

(ii) ONB des IS: $\{\underline{e}_{I1}, \underline{e}_{I2}, \underline{e}_{I3}\}$ ... raumfest

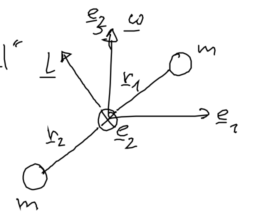
$$\underline{\underline{\Theta}}(t) = \Theta_{Iij}(t) \, \underline{e}_{Ii} \otimes \underline{e}_{Ij}, \quad \Theta_{Iij}(t) = \underline{e}_{Ii} \cdot \underline{\underline{\Theta}}(t) \, \underline{e}_{Ij} \quad (10.28)$$

... zeitabh.

- kontinuierliche Masseverteilung

$$\underline{\underline{\Theta}} \underset{(10.3)}{=} \int \underbrace{d^3r \, \rho(\underline{r})}_{dm} \left[|\underline{r}|^2 \underline{\underline{1}} - \underline{r} \otimes \underline{r} \right] \quad (10.29)$$

- Bsp: „Hantel"



$$\underline{r}_1 = \frac{d}{\sqrt{2}} (\underline{e}_1 + \underline{e}_3), \quad \underline{r}_2 = -\frac{d}{\sqrt{2}} (\underline{e}_1 + \underline{e}_3), \quad |\underline{r}_i| = d$$

$$\Theta_{11} = \Theta_{33} = m d^2 \, 2 \left(1 - \tfrac{1}{2}\right) = m d^2$$

$$\Theta_{13} = \Theta_{31} = m d^2 \, 2 \left(0 - \tfrac{1}{2}\right) = -m d^2$$

$$\Theta_{22} = m d^2 \, 2 \cdot (1 - 0) = 2 m d^2$$

$$\Theta_{ij} = 0, \text{ sonst}$$

$$\longrightarrow \underline{\underline{\Theta}} = m d^2 \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix} \quad (10.30)$$

$\underline{\omega = \omega \underline{e}_3}$: $\quad \underline{L} = \underline{\underline{\Theta}} \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} = m d^2 \omega \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$\underline{L} \nparallel \underline{\omega} \rightarrow \frac{d\underline{L}}{dt} \neq 0 \longleftrightarrow$ Lagerkräfte

e) Kinetische Energie: $T = \frac{1}{2} \sum_\nu m_\nu \dot{\underline{r}}_{I\nu}^2$

mit $\dot{\underline{r}}_{I\nu} = \dot{\underline{R}} + \underline{\omega} \times \underline{r}_\nu$

(i) $\underline{R} = \underline{R}_S$: $\xrightarrow[\sum_\nu m_\nu \underline{r}_\nu = 0]{}$

$$\boxed{\begin{aligned} T &= T_{trans} + T_{rot} \\ &= \frac{1}{2} M \dot{\underline{R}}_S^2 + \frac{1}{2} \sum_\nu m_\nu (\underline{\omega} \times \underline{r}_\nu)^2 \end{aligned}} \quad (10.31)$$

Untersuche: $\underline{\omega} \cdot \underline{L} \underset{(10.20)}{=} \sum_\nu m_\nu \underbrace{\underline{\omega} \cdot [\underline{r}_\nu \times (\underline{\omega} \times \underline{r}_\nu)]}_{} = \sum_\nu m_\nu (\underline{\omega} \times \underline{r}_\nu)^2$

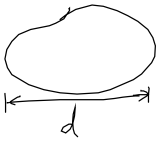
$\underline{a} \cdot (\underline{b} \times \underline{c}) = \underline{c} \cdot (\underline{a} \times \underline{b})$

$$\longrightarrow \boxed{T_{rot} = \frac{1}{2} \underline{\omega} \cdot \underline{L} = \frac{1}{2} \underline{\omega} \cdot \underline{\underline{\Theta}}\, \underline{\omega} = \frac{1}{2} \omega_i \Theta_{ij} \omega_j} \quad (10.32)$$

... für Rotation um Aufpunkt

(ii) $\dot{\underline{R}} = 0$: $\boxed{T = T_{rot}}$

f) potentielle Energie: $U(\underline{r}_{I\nu})$ für m_ν, $\nu = 1, \dots N$



$\underline{r}_{I\nu} = \underline{R} + \underline{r}_\nu$ mit $|\underline{r}_\nu| \leq d$

$U(\underline{r}_{I\nu}) \overset{\text{Taylor}}{=} U(\underline{R}) + \underline{r}_\nu \cdot \text{grad}_\nu U(\underline{r}_{I\nu})\big|_R$

falls $d\, |\text{grad}_\nu U(\underline{r}_{I\nu})|_R| \ll U(\underline{R})$

$$\longrightarrow \boxed{U(\underline{r}_{I\nu}) \approx U(\underline{R})}$$

g) Eigenschaften von $\underline{\underline{\Theta}}$

(1) Hauptachsentrafo:

• Eigenwertgl.:

$$\boxed{\begin{aligned} &\underline{\underline{\Theta}}\, \underline{e}^{(i)} = \Theta_i\, \underline{e}^{(i)} \\ &\left.\begin{aligned} &(\text{o.B.})\ \Theta_i \geq 0 \ \dots \text{ Hauptträgheitsmomente} \\ &\{\underline{e}^{(1)}, \underline{e}^{(2)}, \underline{e}^{(3)}\} \dots \text{ Hauptachsensystem} \end{aligned}\right\} \text{ von } \underline{\underline{\Theta}} \\ &\qquad \text{mit } \underline{e}^{(i)} \cdot \underline{e}^{(j)} = \delta_{ij} \end{aligned}} \quad (10.35)$$

• Diagonalgestalt von $\underline{\underline{\Theta}}$: (10.36)

$$\tilde{\Theta}_{ij} = \underline{e}^{(i)} \cdot \underbrace{\underline{\underline{\Theta}}\, \underline{e}^{(j)}}_{\Theta_j \underline{e}^{(j)}} = \underbrace{\Theta_j\, \delta_{ij}}_{\text{keine Sum konv.}} \longrightarrow \tilde{\underline{\underline{\Theta}}} = \begin{pmatrix} \Theta_1 & & 0 \\ & \Theta_2 & \\ 0 & & \Theta_3 \end{pmatrix}$$

also: $\underline{\underline{\Theta}} = \sum_{i=1}^{3} \Theta_i \underbrace{\underline{e}^{(i)} \otimes \underline{e}^{(i)}}_{}$ (10.37)

3EW 3 Eulerwinkel $\longrightarrow$ 6 unabh. Komp. von Θ

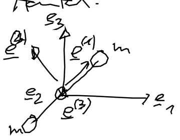
• Fälle: (i) $\Theta_1 \neq \Theta_2 \neq \Theta_3$: unsymmetr. Kreisel

(ii) $\Theta_1 = \Theta_2 \neq \Theta_3$: achsensymm. "

(iii) $\Theta_1 = \Theta_2 = \Theta_3$: Würfel oder Kugel

• physikal.: $\underline{\omega} \parallel \underline{e}^{(i)} \longrightarrow \underline{L} \parallel \underline{\omega}$ ... stabile Drehrichtung!

• Bsp: Hantel:



$$\underline{\underline{\Theta}} = md^2 \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$\underline{e}^{(1)} = \frac{1}{\sqrt{2}}(\underline{e}_1 + \underline{e}_3) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \ \Theta_1 = 0$$

$$\underline{e}^{(2)} = -\frac{1}{\sqrt{2}}(-\underline{e}_1 + \underline{e}_3) = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \ \Theta_2 = 2md^2$$

$$\underline{e}^{(3)} = \underline{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ \Theta_3 = 2md^2$$

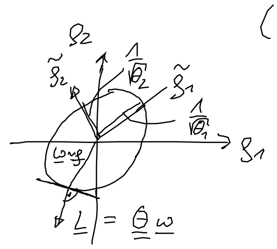
(2) Trägheitsellipsoid = Fläche der Rot.energie

• $T = \frac{1}{2}\, \underline{\omega} \cdot \underline{\underline{\Theta}}\, \underline{\omega}$ mit $\underline{\varrho} = \frac{\underline{\omega}}{\sqrt{2T}}$

$$\longrightarrow \boxed{1 = \varrho_i \Theta_{ij} \varrho_j \xrightarrow[\text{trafo}]{\text{Hauptachse}} 1 = \Theta_1 \tilde{\varrho}_1^{\,2} + \Theta_2 \tilde{\varrho}_2^{\,2} + \Theta_3 \tilde{\varrho}_3^{\,2}} \quad (10.38)$$

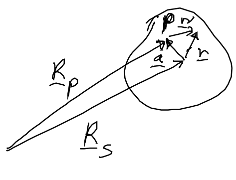
... Trägheitsellipsoid mit Halbachsen $\frac{1}{\sqrt{\Theta_i}}$

(Flächen 2. Grades)



Poinsotsche Konstruktion ... Beweis: Übung

(3) Satz von Steiner



$\underline{\underline{\Theta}}^{(s)}$ bezogen auf S bekannt

$$\longrightarrow \boxed{\underline{\underline{\Theta}}^{(P)} = M\left(|\underline{a}|^2 \mathbb{1} - \underline{a} \otimes \underline{a}\right) + \underline{\underline{\Theta}}^{(s)}} \quad (10.39)$$

... Trägheitstensor bzgl. Punkt P

Gesamtmasse

Beweis: Übung