Lecture 15 summary

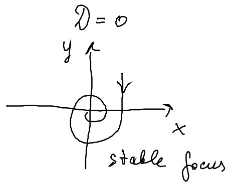
3. Noise-induced dynamics

– systems without ext. periodic forcing

– deterministic system has a stable fixed point

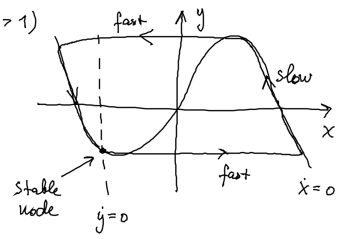
1. Example : van der Pol system below Hopf bifurcation ($\varepsilon < 0$)

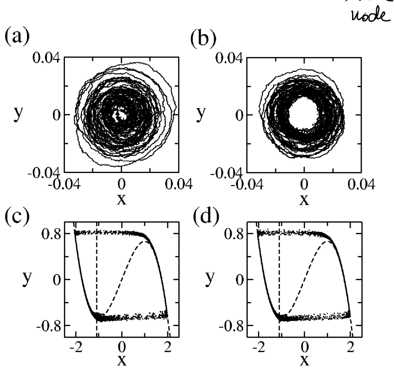
$$\boxed{\begin{aligned} \dot{x} &= y \\ \dot{y} &= \underbrace{(\varepsilon - x^2)\,y}_{\text{nonlinear damping}} - \omega_0^2 x + \sqrt{2D}\,\xi(t) \end{aligned}}$$



2. Example : FitzHugh–Nagumo model in excitable regime ($a > 1$)

$$\boxed{\begin{aligned} \varepsilon \dot{x} &= x - \frac{x^3}{3} - y \\ \dot{y} &= x + a + \sqrt{2D}\,\xi(t) \end{aligned}}$$





van der Pol system
$D = 0.003$

FitzHugh–Nagumo system
$D = 0.09$

$D \neq 0$: noise-induced oscillations in FHN model

Janson, Balanov, Schöll : PRL 93, 010601 (2004)

Balanov, Janson, Schöll : Physica D 199, 1 (2004)

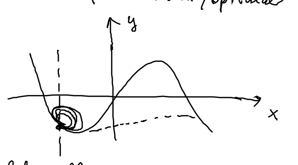
Schöll, Balanov, Janson, Neiman : Stoch. Dyn. 5, 281 (2005)

3.3 Coherence resonance (CR)

– constructive role of noise

CR : the best regularity (coherence) of noise-induced oscillations is achieved for interm./optimal noise intensity $D_{opt.}$

FHN

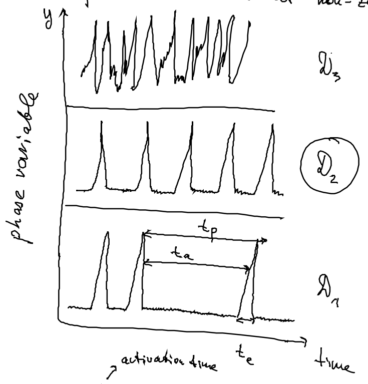


subthreshold osillations

Two competing time scales with opposite dependence upon noise intensity $\rightarrow$

non-monotonic dependence of coherence upon noise intensity $\rightarrow$

optimum coherence at non-zero noise intensity.



$D_3 > D_2 > D_1$

$$t_p = t_a + t_e$$

t_p: pulse duration; t_a: activation time; t_e: excursion time

t_a rapidly decreases with noise intensity

t_e weakly depends on noise

Measures of coherence resonance

- normalized standard deviation of interspike interval (ISI)
  (normalized fluctuations of the pulse durations)

$$R_T = R = \frac{\sqrt{\langle t_p^2 \rangle - \langle t_p \rangle^2}}{\langle t_p \rangle} = \frac{\sqrt{\mathrm{var}(t_p)}}{\langle t_p \rangle}$$

Pikovsky, Kurths, PRL 78, 775 (1997) CR in FHN, the name CR

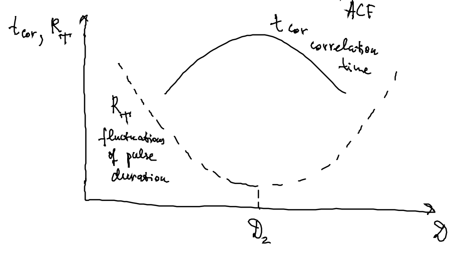
Gang, Ditzinger, Ning, Haken, PRL 71, 807 (1993)
"Stochastic resonance without external periodic forcing"

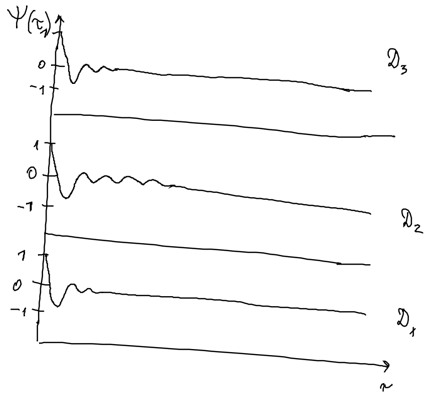
R_T can be used in the case of spiking dynamics
(not used for non-excitable systems since there are no spikes)

- correlation time

$$t_{cor} = \frac{1}{\Psi(0)} \int_0^\infty |\Psi(s)|\, ds$$

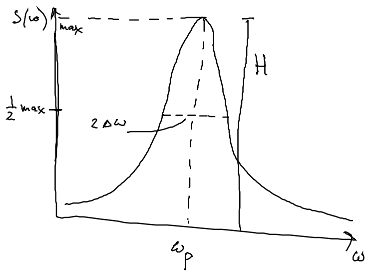
Ψ: ACF





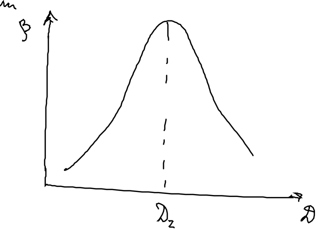
- signal-to-noise ratio (SNR)
  of the spectral peak height H and its relative width

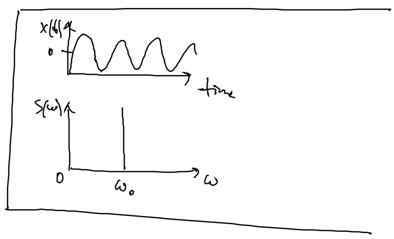
$$\beta = \frac{H}{\Delta\omega/\omega_p}$$



HWHM
half width at half maximum $\rightarrow \Delta\omega$

FWHM
full width at half maximum $\rightarrow 2\Delta\omega$





Correlation time

$$t_{cor} = \frac{1}{\Psi(0)} \int_0^\infty |\Psi(s)|\, ds$$

$$\text{ACF}: \Psi(s) = \langle [x(t+s) - \langle x \rangle][x(t) - \langle x \rangle] \rangle$$

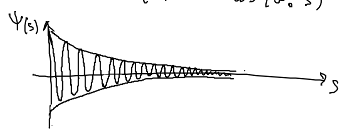
$$\Psi(0) = \sigma^2$$

Motivation of definition

for linear stoch. processes $\dot{x} = -(k + i\omega_0)x + \xi(t)$

(Ornstein-Uhlenbeck proc.
$\operatorname{Re} e^{-(k+i\omega_0)s}$)

$$\Psi(s) = \Psi(0)\, e^{-ks} \cos(\omega_0 s)$$



exp.-ly decaying envelope

carrier frequency ω_0

$k > 0$ (stable fixed point)

Relation between k and t_{cor}

$$t_{cor} = \int_0^\infty e^{-ks} |\cos \omega_0 s|\, ds$$

approx. $\frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \cos\varphi\, d\varphi = \frac{2}{\pi}$ for $k \ll \omega_0$ (filling factor)

$\Rightarrow t_{cor} \approx \frac{2}{\pi} \int_0^\infty e^{-ks} ds = \frac{2}{\pi k}$

$t_{cor} \approx \frac{2}{\pi k}$

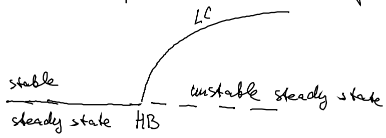
$-k = \frac{-2}{\pi t_{cor}}$

therefore $\Psi(s) = \Psi(0)\, e^{-\frac{2}{\pi}\frac{s}{t_{cor}}} \cos(\omega_0 s)$

$K = |Re$ (eigenvalue of the fixed point)$|$

$Re\,\lambda < 0$

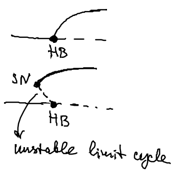
bifurcation parameter (distance from HB)



The more stable is the fixed point, the shorter is correlation time (further away from HB)

Hopf bifurcation → supercritical case

HB ↘ subcritical case



SN – saddle-node bifurcation of limit cycles

between SN and HB → bistability: stable fixed point and stable limit cycle