Lecture 2 summary

Moment of probability distribution

$M_\vartheta = \langle x^\vartheta \rangle$

Moment generating function

$$Z(\alpha) = \langle e^{\alpha x} \rangle = \sum_{\vartheta=0}^{\infty} \frac{\alpha^\vartheta}{\vartheta!} M_\vartheta$$

$\alpha = is$: inverse Fourier transform of

$$Z(is) = \int dx\, \rho(x)\, e^{isx}$$

$$\rho(x) = \frac{1}{2\pi} \int ds\, Z(is)\, e^{-ixs}$$

Cumulant generating function

$$\Gamma(\alpha) = \ln \langle e^{\alpha x} \rangle = \sum_{\vartheta=0}^{\infty} \frac{\alpha^\vartheta}{\vartheta!} C_\vartheta$$

Cumulants are additive for uncorrelated random variables

$$\langle (x_1 + x_1)^\vartheta \rangle_c = \langle x_1^\vartheta \rangle_c + \langle x_2^\vartheta \rangle_c$$

(not valid for moments. Moments factorize

$$\langle (x_1 + x_2)^2 \rangle = \langle x_1^2 \rangle + \langle x_2^2 \rangle + 2 \langle x_1 \rangle \langle x_2 \rangle$$

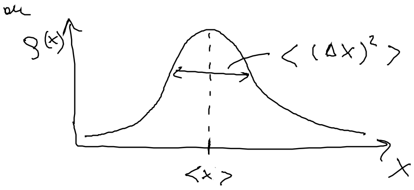
Fluctuation $\Delta x = x - \langle x \rangle$ is deviation from the mean

$$\langle \Delta x \rangle = 0$$

Variance $\langle (\Delta x)^2 \rangle = \langle (x - \langle x \rangle)^2 \rangle = \langle x^2 \rangle - 2 \langle x \rangle \langle x \rangle + \langle x \rangle^2 =$

$$= \langle x^2 \rangle - \langle x \rangle^2$$

Measure for the width of the distribution



Covariance matrix

$$\langle \Delta x_k \Delta x_\ell \rangle = \langle x_k x_\ell \rangle - \langle x_k \rangle \langle x_\ell \rangle$$

non-diagonal elements vanish for uncorrelated random variables

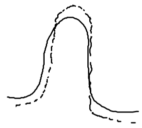
Relation between cumulants and moments

$\langle x \rangle_c = \langle x \rangle$ mean

$\langle x^2 \rangle_c = \langle (\Delta x)^2 \rangle$ variance (width)

$\langle x^3 \rangle_c = \langle (\Delta x)^3 \rangle$ skewness (measure of asymmetry)

$\langle x^4 \rangle_c = \langle (\Delta x)^4 \rangle - 3 \left(\langle \Delta x^2 \rangle \right)^2$ kurtosis



Central limit theorem

Let $x_1, \dots x_n$ be uncorrelated random variables with $\langle x_i \rangle = 0$ mean

$\langle (\Delta x_i)^2 \rangle = b_i^2$ variance

(for example, a random walk = Brownian motion with time step n).

Then the distribution converges from $x = \sum_i x_i$ for $n \to \infty$ to Gaussian (normal) distribution

$$\rho(x) = \frac{1}{\sigma \sqrt{2\pi}} \exp\left(- \frac{(x - \langle x \rangle)^2}{2\sigma^2} \right) \qquad \sigma^2 = \sum_i b_i^2 \quad \text{variance}$$

$$\sigma^2 = \langle x^2 \rangle_c \, ; \quad \langle x^k \rangle_c = 0 \quad f \quad k > 2$$

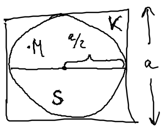
higher order cumulants vanish

Geometric probability

How to find π experimentally?

A particle M which we throw into a squire.

There is a circle inscribed into the squire.



K - squire with a side length a

S - circle;

$K = a^2$; $\quad S = \left(\frac{a}{2}\right)^2 \cdot \pi$

P - probability that the particle M gets into the circle

$$P = \frac{\text{area of a circle}}{\text{area of a square}} = \frac{\frac{\pi}{4} \cancel{a^2}}{\cancel{a^2}} = \frac{\pi}{4}$$

1.2 Markov process

Stochastic process:

time evolution of a random variable $X(t)$

time-dependent probability $p(x_1, t_1; x_2, t_2; x_3, t_3 \dots)$

$x_1, x_2, x_3, \dots$ realizations of $X(t)$

Conditional probability

$P(A|B) = \frac{P(A \cap B)}{P(B)}$

joint probability

$$p(x_1, t_1 \,/\, x_2, t_2; x_3, t_3, \dots) = \frac{p(x_1, t_1; x_2, t_2; \dots)}{p(x_2, t_2; x_3, t_3, \dots)}$$

$(t_1 \geq t_2 \geq t_3 \geq \dots)$

Markov process

$$\boxed{p(x_1, t_1 \,/\, x_2, t_2; x_3, t_3 \dots) = p(x_1, t_1 \,/\, x_2, t_2)}$$

stochastic process "without memory"

not the whole past $(t_2, t_3, t_4, \dots)$ defines the future (t_1),
but only the present (t_2)

"memorylessness"

Therefore: $p(x_1,t_1; x_2,t_2; x_3,t_3; \dots) = p(x_1,t_1 / x_2,t_2)\, p(x_2,t_2; x_3,t_3; \dots)$

$\Rightarrow p(x_1,t_1 / x_2,t_2)\, p(x_2,t_2 / x_3,t_3)\, p(x_3,t_3; \dots)$

$$= p(x_1,t_1 / x_2,t_2)\, p(x_2,t_2 / x_3,t_3) \dots p(x_{n-1},t_{n-1} / x_n,t_n)\, p(x_n,t_n)$$

$$t_1 \leftarrow t_2 \leftarrow t_3 \; \dots \leftarrow t_{n-2} \leftarrow t_n$$

(Markov chain)

In the case of joint probability for uncorrelated events

$$\sum_B P(A \cap B \cap C) = \underbrace{\sum_B P(B)}_{1} P(A \cap C) = P(A \cap C)$$

Therefore,
(also non-Markov)

$$p(x_1,t_1) = \int dx_2\, p(x_1,t_1; x_2,t_2) \overset{\text{def. of cond. prob.}}{=}$$

$$= \int dx_2\, p(x_1,t_1 / x_2,t_2)\, p(x_2,t_2)$$

Simplified notation $\quad p(1) = \int dx_2\, p(1/2)\, p(2) \quad (1)$

$$p(1/3) = \int dx_2\, p(1;2/3) =$$

$$= \int dx_2 \frac{p(1;2;3)}{p(3)} = \int dx_2 \frac{p(1;2;3)}{p(2;3)} \frac{p(2;3)}{p(3)}$$

$$= \int dx_2\, p(1/2;3)\, p(2/3) \quad (2)$$

Now we use Markovian assumption $\quad p(1/2;3) = p(1/2)$

we get $\Rightarrow \quad p(1/3) = \int dx_2\, p(1/2)\, p(2/3) \quad (3)$

$$\boxed{p(x_1,t_1 / x_3,t_3) = \int dx_2\, p(x_1,t_1 / x_2,t_2)\, p(x_2,t_2 / x_3,t_3)}$$

Chapman – Kolmogorov equation (fundamental equation for conditional probabilities of Markov processes)

For discrete events

$$P(n_1,t_1 / n_3,t_3) = \sum_{n_2} P(n_1,t_1 / n_2,t_2)\, P(n_2,t_2 / n_3,t_3)$$

<u>Stationary</u> stochastic process (stat. random process)
is a stochastic process whose joint probability distribution does not change when shifted in time

$$p(x_1, t_1; x_2, t_2; x_3, t_3; \dots) = p(x_1, t_1+\varepsilon; x_2, t_2+\varepsilon; x_3, t_3+\varepsilon; \dots)$$

$\Rightarrow p(x, t) = p(x)$ probability distribution does not depend on time

$\Rightarrow \langle x \rangle$ also does not depend on time

joint prob.

$$p(x_1, t_1; x_2, t_2) = p(x_1, t_1 - t_2; x_2, 0)$$

cond. prob.

$$p(x_1, t_1 | x_2, t_2) = p(x_1, t_1 - t_2 | x_2, 0)$$

(for Markov process)

$\Rightarrow$ <u>Autocorrelation function</u> $\langle x(t)\, x(t+\tau) \rangle = G(\tau) = G(-\tau)$

(correlation of the process with a delayed copy of itself)