Lecture 11 summary

(i) Wiener process (no drift)

SDE $\dot{x} = \sqrt{2D}\,\xi(t)$

FPE $\frac{\partial}{\partial t} p = D \frac{\partial^2}{\partial x^2} p$

- has independent increments
- nonstationary (probability density explicitly depends on time)
- variance grows linearly in time $\langle (\Delta X(t))^2 \rangle = 2D\,(t-t_0)$
- mean value is defined by I.C. $\langle x(t) \rangle = x_0$
- Gaussian process, Markov process

(ii) Ornstein-Uhlenbeck process

SDE $\dot{x} = \underbrace{-kx}_{\text{linear drift}} + \sqrt{2D}\,\xi(t)$

FPE $\frac{\partial}{\partial t} p = \frac{\partial}{\partial x}(kxp) + D \frac{\partial^2}{\partial x^2} p$

- stationary
- variance is bounded $\langle \Delta X^2 \rangle = \frac{D}{k}$
- Gaussian process, Markov process
- ACF depends only on time differences

$$\langle X(t+\tau)\,X(t) \rangle = \frac{D}{k} e^{-k|\tau|} = D\,\tau_c\, e^{-|\tau|/\tau_c}$$

$\tau_c = \frac{1}{k}$ correlation time

3. Noise-induced oscillations and patterns,

noise → undesired, destructive, makes deterministic dynamics irregular

Noisy sources in nonlinear dynamical systems → induce new regime
→ more ordered behaviour
→ formation of more regular temporal dynamics (increase of degree of coherence)

Constructive role of noise
↙ stochastic resonance
↘ coherence resonance

3.1 Stochastic resonance (SR)

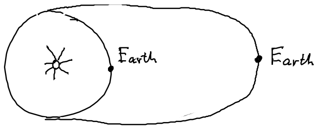
Enhancement of the weak periodic signal with the help of noise.

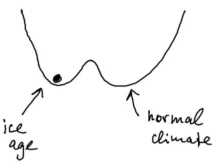
The term SR was introduced in 1981 – 1982

[Benzi et al. J. Phys. A 14 (1981)
Benzi et al. Tellus 34 (1982), p. 10
Nicolis, Tellus 34 (1982), p. 1-9]

Example 1. Periodicity of the ice age on the Earth

a model to explain the almost periodic recurrences (repetitions) of the Earth's ice ages.





symmetric double-well potential

A particle in a double-well potential driven by a periodic force.
Periodic force → tiny oscillations of the eccentricity of Earth's orbit: period ~ 10^5 years; the orbit changes from the nearly circular to more elliptical.

The switching was achieved by adding random force into the model
↓ fluctuations of the atmosphere

The fundamental result: temporally ordered transitions; the climate almost followed the vanishing small external periodic perturbation for some finite noise strength of the atmosphere.

Stochastic resonance
- nonlinear system
- periodic force
- noise

SR is a mechanism by which a nonlinear system in a noisy environment gets an inhanced sensitivity towards small external periodic force for an interm./optimal noise intensity.

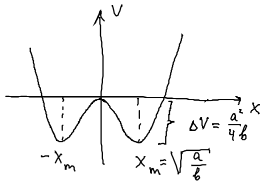
General scheme of SR

$$\xrightarrow{s(t)} \boxed{\frac{dx}{dt} = f(x, s(t), D)} \xrightarrow{x(t)}$$

The response of the nonlinear system to weak input signal $s(t)$ is significantly increased for some optimal non-zero noise.

Example 2. Overdamped Brownian particle in bistable potential

$V(x) = -\frac{a}{2}x^2 + \frac{b}{4}x^4$; periodic force $A_0 \cos(\Omega t)$; white noise $\xi(t)$



Langevin equation (SDE)

$$\boxed{\dot{x} = -V'(x) + A_0 \cos(\Omega t) + \xi(t)}$$

$\langle \xi(t) \rangle = 0$

$\langle \xi(t)\, \xi(t') \rangle = 2D\, \delta(t - t')$

Overdamped

The motion of a particle in a potential field $U(y)$ in the presence of damping is given by

$$\ddot{y} + \underbrace{\gamma \dot{y}}_{\text{damping}} = -U'(y)$$

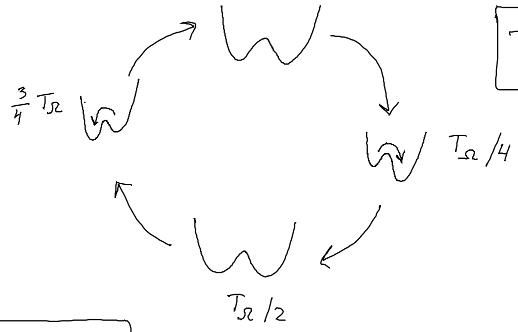
overdamped → $\gamma \gg 1$ ⇒ we neglect second derivative

$a = b = 1 \Rightarrow \quad \dot{x} = x - x^3 + A_0 \cos(\Omega t) + \xi(t)$

Noise-induced switchings between $-x_m$ and x_m <u>without periodic force</u> are described by Kramers rate:

$$r_k = \frac{1}{T_k(D)} = \frac{1}{2\pi} \exp\left(-\frac{\Delta V}{D}\right) \qquad (a = b = 1)$$

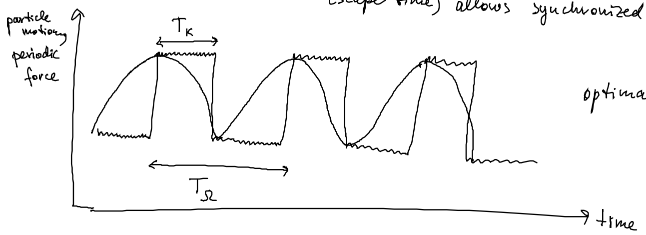
Periodic driving: the double-well potential $V(x,t) = V(x) - A_0 x \cos(\Omega t)$ is tilted back and forth $\rightarrow$ raising and lowering the potential barriers of the right and left well.



$$\boxed{T_\Omega = \frac{2\pi}{\Omega}}$$

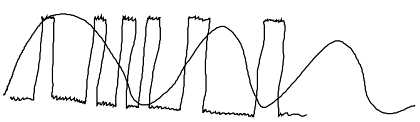
$$\boxed{T_\Omega \approx 2 T_k(D)}$$

A suitable dose of noise (when the period of driving approx-ly equals twice the noise-induced escape time) allows synchronized hopping.

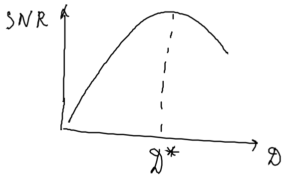


optimal noise D^*

D is too small

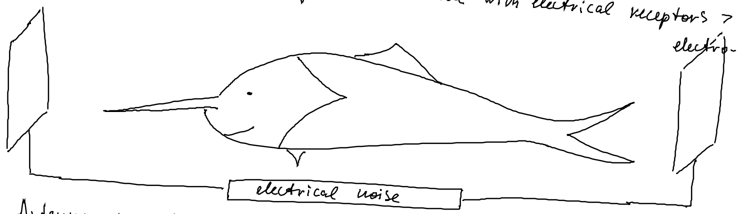


D is too large



Examples: Brownian particles, lasers, tunnel diodes, quantum systems, chemical systems, sociological and biological systems (music as noise for mental activity).

Example 3. Paddlefish: in North America the rivers are not clear →
→ the fish has developed an antenna with electrical receptors > 10.000 electro-receptors



Antenna is used instead of eyes to catch the food (plancton).
Plancton emits signal of low frequency.

[Russel, Wilkens and Moss, Nature 1999]

In the experiment external random electrical field was applied and the amount of plancton was measured.

- low noise: a certain amount of food
- intermediate noise: the largest amount of food
- large noise: can not find plancton