Lecture 18 summary

Stochastic bifurcations

P-bifurcations: change in probability distribution

$\mathcal{D}$ - bifurcation: change of stability

3.4.3 Coherence resonance in Stuart-Landau oscillator

$$\dot{z} = -i\omega_0 z + z F(z) + \sqrt{2\mathcal{D}}\,\xi(t)$$

supercritical HB : $F(z) = a_1 - |z|^2$

subcritical HB : $F(z) = a_2 + |z|^2 - |z|^4$

CR occurs in a strict sense only in the subcritical case: noise improves, indeed, the temporal coherence of oscillations since $\Delta\omega$ demonstrates a minimum

3.4.4 CR in Duffing-van der Pol oscillator

$$\ddot{x} - (\varepsilon + x^2 - x^4)\dot{x} + x + \beta x^3 = \sqrt{2\mathcal{D}}\,n(t), \quad \beta \geq 0$$

new variables : $a(t)$ and $\varphi(t)$

$$\boxed{\begin{aligned} \dot{a} &= \left(\frac{\varepsilon}{2} + \frac{a^2}{8} - \frac{a^4}{16}\right) a + \frac{\mathcal{D}}{2a} + \sqrt{\mathcal{D}}\,n_1(t) \\ \dot{\varphi} &= \frac{3\beta}{8} a^2 + \frac{\sqrt{\mathcal{D}}}{a} n_2(t) \end{aligned}}$$

averaged equations for slow variables

$$\dot{a} = \left(\frac{\varepsilon}{2} + \frac{a^2}{8} - \frac{a^4}{16}\right) a + \frac{\mathcal{D}}{2a} + \sqrt{\mathcal{D}}\,n_1(t)$$

The stochastic process described by this equation is characterized by the drift coefficient

$$A(a) = \left(\frac{\varepsilon}{2} + \frac{a^2}{8} - \frac{a^4}{16}\right) a + \frac{\mathcal{D}}{2a}$$

and by diffusion coefficient $B = \frac{1}{2} b^2 = \frac{D}{2}$

L 10

2.3 Langevin eq.

$$\frac{dx}{dt} = a(x,t) + b(x,t)\xi(t)$$

FPE

$$\frac{\partial}{\partial t} p(x,t) = -\frac{\partial}{\partial x}\left[a(x,t)\, p(x,t)\right] + \\ + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[\left(b(x,t)\right)^2 p(x,t)\right]$$

$a = A$ - drift

$\frac{b^2}{2}$ - diffusion

$$\frac{\partial p(a,t)}{\partial t} = -\frac{\partial}{\partial a}\left[A(a)\, p(a,t)\right] + \underbrace{\frac{b^2}{2}}_{=B} \frac{\partial^2}{\partial a^2} p(a,t) \qquad \text{FPE}$$

We substitute drift and diffusion coef. into FPE

$$\boxed{\frac{\partial p(a,t)}{\partial t} = -\frac{\partial}{\partial a}\left[\left(\frac{\varepsilon a}{2} + \frac{a^3}{8} - \frac{a^5}{16} + \frac{D}{2a}\right) p(a,t)\right] + \frac{D}{2}\frac{\partial^2 p(a,t)}{\partial a^2}}$$

The stationary solution of FPE for one-dimensional stochastic process $a(t)$:

$$p^*(a) = \frac{C_0}{B(a)} \exp\left[\int \frac{A(a)}{B(a)}\, da\right],$$

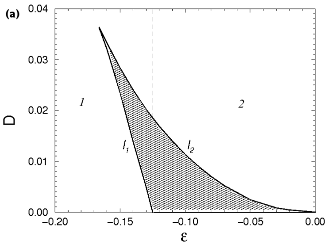
where $C_0 = \text{const}$.

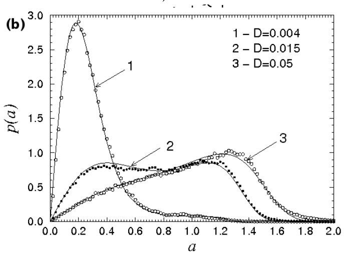
Here we get: $p^*(a) = N \cdot a \cdot \exp\left[-\frac{1}{48D} a^2 \left(a^4 - 3a^2 - 24\varepsilon\right)\right]$,

where N is normalization constant (we can find it numerically).

The shape of the distribution is the same for isochronous and anisochronous oscillations.

Stochastic bifurcations (P-bif.): amplitude probability distribution changes → we can analyze analytically the maxima of $p(a)$.

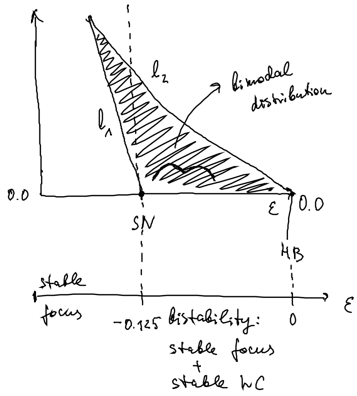




probability distributions

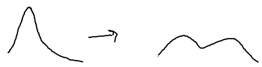
$\varepsilon = -0.13$

(for $D=0$ we only have one attractor → stable focus)



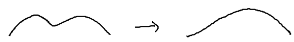
$D = 0$

l_1:



stochastic P-bifurcation

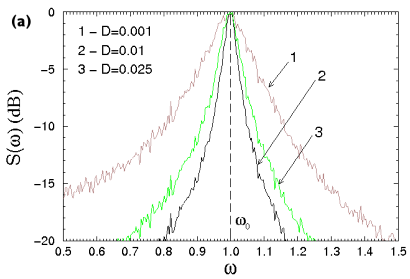
l_2:



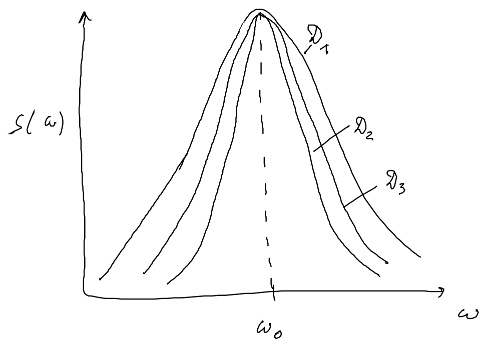
stochastic P-bifurcation

In the shaded region the stationary amplitude distribution is bimodal. For $D=0$ the region of bistability is in the regime between HB ($\varepsilon=0$) and SN ($\varepsilon=-0.125$).

In the precence of noise the bimodality shifts towards smaller values of ε and becomes narrower.



$\varepsilon = -0.13$

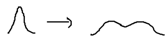


D_1, D_2, D_3 – we tune noise intensity

The spectral line width becomes minimal for an intermediate/optimal noise intensity ⇒
⇒ Coherence resonance occurs.

$\beta = 0$!

The effect of CR is connected to stochastic bifurcations $\rightarrow$
$\rightarrow$ CR is the most pronounced for noise intensity within a region of a bimodal amplitude distribution.
$\Rightarrow$ CR occurs after the stochastic P-bifurcation on line ℓ_1:



Example of synthetic gene oscillator

$$\dot{x}(t) = \frac{1 + x^2 + \alpha \sigma x^4}{(1 + x^2 + \sigma x^4)(1 + y^4)} - \gamma_x x + \sqrt{2D}\, n(t)$$

$$\tau_y \dot{y}(t) = \frac{1 + x^2 + \alpha \sigma x^4}{(1 + x^2 + \sigma x^4)(1 + y^4)} - \gamma_y y$$

This dimensionless system describes evolution of concentrations of two proteins x (cI) and y (lac);

τ_y – time scale for y

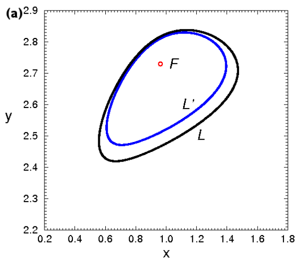
α – degree to which the transcription rate is increased

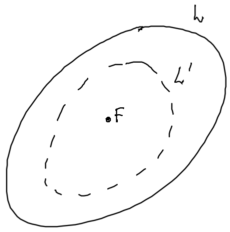
σ – affinity for a dimer

γ_x and γ_y characterize degradation rates

$n(t)$ – Gaussian white noise.

$D = 0$

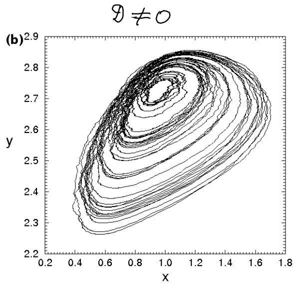


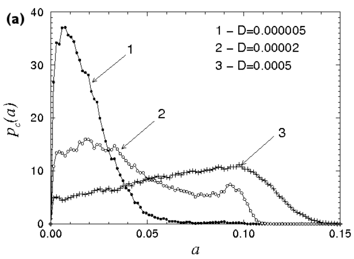


subcritical HB

bistability region:
limit cycle L is stable,
focus F is stable,
limit cycle L' is unstable.

$D \neq 0$





Numerical results

$a = \sqrt{x^2 + y^2}$