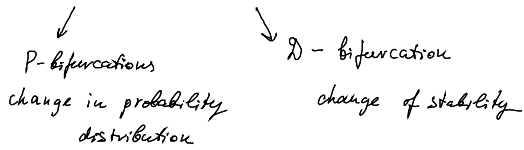


lecture 18 summary

Stochastic bifurcations



3.4.3 Coherence resonance in Stuart-Landau oscillator

$$\dot{z} = -i\omega_0 z + z F(z) + \sqrt{2D} \xi(t)$$

supercritical HB : $F(z) = a_1 - |z|^2$

subcritical HB : $F(z) = a_2 + |z|^2 - |z|^4$

CR occurs in a strict sense only in the subcritical case:
 noise improves, indeed, the temporal coherence of oscillations since $\Delta\omega$ demonstrates a minimum



3.4.4 CR in Duffing-van der Pol oscillator

$$\ddot{x} - (\varepsilon + x^2 - x^4)\dot{x} + x + \beta x^3 = \sqrt{2D} n(t), \quad \beta \geq 0$$

new variables : $a(t)$ and $\varphi(t)$

$$\begin{aligned} \dot{a} &= \left(\frac{\varepsilon}{2} + \frac{a^2}{8} - \frac{a^4}{16} \right) a + \frac{D}{2a} + \sqrt{D} n_1(t) \\ \dot{\varphi} &= \frac{3\beta}{8} a^2 + \frac{\sqrt{D}}{a} n_2(t) \end{aligned}$$

averaged equations for slow variables

$$\dot{a} = \left(\frac{\varepsilon}{2} + \frac{a^2}{8} - \frac{a^4}{16} \right) a + \frac{D}{2a} + \sqrt{D} n_1(t)$$

The stochastic process described by this equation is characterized by the drift coefficient

$$A(a) = \left(\frac{\varepsilon}{2} + \frac{a^2}{8} - \frac{a^4}{16} \right) a + \frac{D}{2a}$$

and by diffusion coefficient $B = \frac{1}{2} b^2 = \frac{D}{2}$

L 10

2.3 Langevin eq.

$$\frac{dx}{dt} = a(x,t) + b(x,t) \xi(t)$$

FPE

$$\frac{\partial}{\partial t} p(x,t) = -\frac{\partial}{\partial x} [a(x,t) p(x,t)] + \frac{1}{2} \frac{\partial^2}{\partial x^2} [b(x,t)^2 p(x,t)]$$

$a = A$ - drift

$\frac{b^2}{2}$ - diffusion

$$\frac{\partial p(a,t)}{\partial t} = -\frac{\partial}{\partial a} [A(a) p(a,t)] + \left(\frac{b^2}{2}\right) \frac{\partial^2}{\partial a^2} p(a,t) \quad \text{FPE}$$

We substitute drift and diffusion coef. into FPE

$$\frac{\partial p(a,t)}{\partial t} = -\frac{\partial}{\partial a} \left[\left(\frac{\varepsilon a}{2} + \frac{a^3}{8} - \frac{a^5}{16} + \frac{D}{2a} \right) p(a,t) \right] + \frac{D}{2} \frac{\partial^2 p(a,t)}{\partial a^2}$$

The stationary solution of FPE for one-dimensional stochastic process $a(t)$:

$$p^*(a) = \frac{C_0}{B(a)} \exp \left[\int \frac{A(a)}{B(a)} da \right],$$

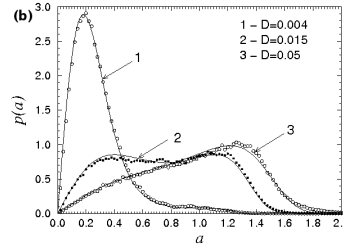
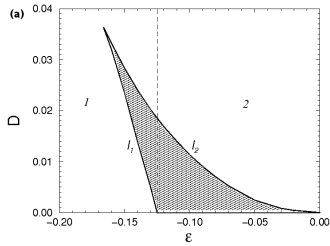
where $C_0 = \text{const.}$

Here we get: $p^*(a) = N \cdot a \cdot \exp \left[-\frac{1}{48D} a^2 (a^4 - 3a^2 - 24\varepsilon) \right],$

where N is normalization constant (we can find it numerically).

The shape of the distribution is the same for isochronous and anisochronous oscillations.

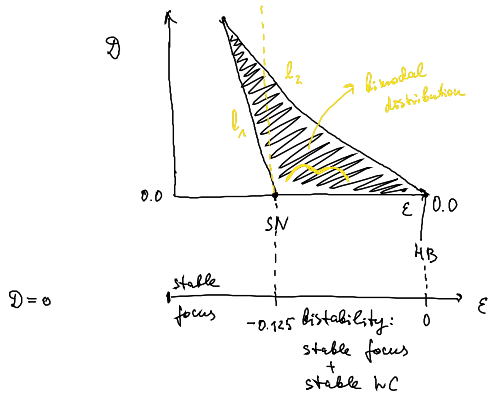
Stochastic bifurcations (P-brf): amplitude probability distribution changes \rightarrow we can analyze analytically the maxima of $p(a)$.

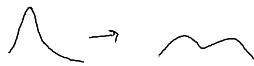



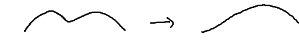
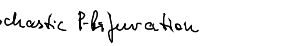
probability distributions

$$\varepsilon = -0.15$$

(for $D=0$ we only have one attractor \rightarrow stable focus)

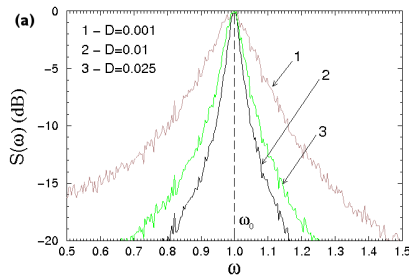


l_1 :  \rightarrow 
stochastic P-bifurcation

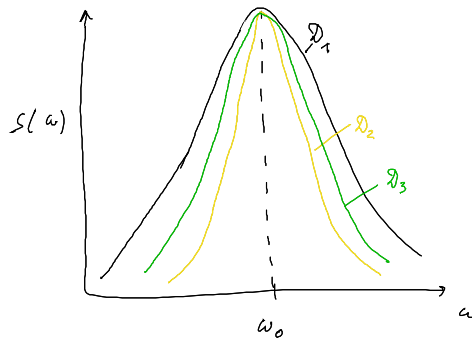
l_2 :  \rightarrow 
stochastic P-bifurcation

In the shaded region the stationary amplitude distribution is bimodal. For $D=0$ the region of bistability is in the regime between HB ($\varepsilon=0$) and SN ($\varepsilon=-0.125$).

In the presence of noise the bimodality shifts towards smaller values of ε and becomes narrower.



$$\varepsilon = -0.13$$



D_1, D_2, D_3 - we tune noise intensity

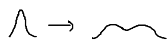
The spectral line width becomes minimal for an intermediate/optimal noise intensity \Rightarrow Coherence resonance occurs.

$\beta = 0!$

The effect of CR is connected to stochastic bifurcations \rightarrow

\rightarrow CR is the most pronounced for noise intensity within a region of a bimodal amplitude distribution.

\Rightarrow CR occurs after the stochastic P-bifurcation on line L_1 :



Example of synthetic gene oscillator

$$\dot{x}(t) = \frac{1+x^2+d\sigma x^4}{(1+x^2+\sigma x^4)(1+y^4)} - \gamma_x x + \sqrt{2\beta} n(t)$$

$$\tau_y \dot{y}(t) = \frac{1+x^2+d\sigma x^4}{(1+x^2+\sigma x^4)(1+y^4)} - \gamma_y y$$

This dimensionless system describes evolution of concentrations of two proteins x (cI) and y (lac);

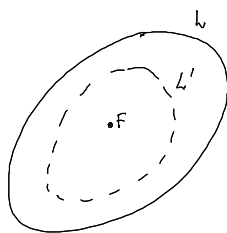
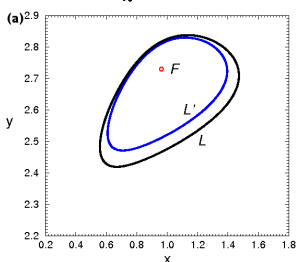
τ_y - time scale for y

d - degree to which the transcription rate is increased

σ - affinity for a dimer

γ_x and γ_y characterize degradation rates

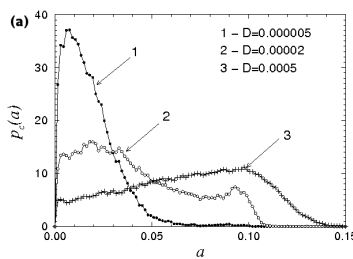
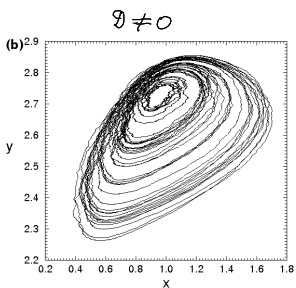
$n(t)$ - Gaussian white noise.
 $\beta = 0$



subcritical HB

stability region:

limit cycle L is stable,
focus F is stable,
limit cycle L' is unstable.



Numerical results

$$\alpha = \sqrt{x^2 + y^2}$$

