Lecture 3 summary

Cumulants and moments

$\langle X \rangle_c = \langle X \rangle$ mean

$\langle X^2 \rangle_c = \langle (\Delta X)^2 \rangle$ variance

$\langle X^3 \rangle_c = \langle (\Delta X)^3 \rangle$ skewness

Stochastic process

random variable $X(t)$ with probability
$p(x_1,t_1;\, x_2,t_2;\, x_3,t_3;\dots)$

Markov process: $p(x_1,t_1 / x_2,t_2;\, x_3,t_3;\dots) =$
$= p(x_1,t_1/x_2,t_2)$

$(t_1 \geq t_2 \geq t_3 \geq \dots)$

Chapman-Kolmogorov equation

$$p(x_1,t_1/x_3,t_3) = \int dx_2\, p(x_1,t_1/x_2,t_2)\, p(x_2,t_2/x_3,t_3)$$

$$p(1/3) = \int dx_2\, p(1/2)\, p(2/3)$$

Ergodicity

For stationary process: ensemble average $\overset{!}{=}$ time average

Time average $\bar{X}(T) = \frac{1}{2T}\int_{-T}^{T} dt\, X(t)$, $T \to \infty$

$$\bar{X}(T) = \langle x \rangle$$

$\bar{X}$ - time average

$\langle x \rangle$ - ensemble average

$\Rightarrow$ calculation of autocorrelation function using time average:

$$G(\tau) = \langle x(t)\,x(t+\tau)\rangle \overset{!}{=} \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} dt\, x(t)\,x(t+\tau)$$

for ergodic proc.

Relation to spectral properties

Fourier transform:

$$\hat{x}(\omega,T) = \frac{1}{2\pi}\int_{-T}^{T} dt\, e^{i\omega t}\, x(t)$$

$$G(\tau) = G(-\tau)$$

Power spectral density

$$S(\omega) = \lim_{T\to\infty} \frac{\pi}{T}\,|\hat{x}(\omega_j,T)|^2 = \lim_{T\to\infty}\frac{\pi}{T}\frac{1}{(2\pi)^2}\int_{-T}^{T} dt \int_{-T}^{T} dt'\, e^{i\omega \overbrace{(t-t')}^{-\tau}}\, x(t)\, x(\underset{t+\tau}{t'}) =$$

$$= \lim_{T\to\infty} \frac{1}{2\pi}\int d\tau\, e^{-i\omega\tau}\, \underbrace{\frac{1}{2T}\int_{-T}^{T} dt\, x(t)\,x(t+\tau)}_{G(\tau)}$$

$$\boxed{S(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\tau\, e^{-i\omega\tau}\, G(\tau)}$$ Wiener-Khinchin theorem

This theorem establishes the relation between autocorrelation function and power spectral density (power spectrum)

Inverse $G(\tau) = \int_{-\infty}^{\infty} d\omega\, e^{i\omega\tau}\, S(\omega)$

<u>homogeneous</u> stochastic process

$\lim_{t\to\infty} p(x,t/\underbrace{x',0}_{\text{initial conditions}}) = p(x)$ (stationary process is reached from any I.C.)

initial conditions = I.C.

1.3 Differential Chapman-Kolmogorov equation

From Chapman-Kolmogorov eq. (discrete in t)

$$p(x_1,t_1/x_3,t_3) = \int dx_2\, p(x_1,t_1/x_2,t_2)\, p(x_2,t_2/x_3,t_3)$$
$$(t_1 \geq t_2 \geq t_3 \ldots)$$

one can derive a diff. eq. for $p(x,t/x_0,t_0)$:

Assumptions: for all $\varepsilon > 0$

(i) $\lim_{\Delta t\to 0} \frac{p(x,t+\Delta t/z,t)}{\Delta t} = \underbrace{W(x/z,t)}$ uniformly in z,x,t for $|x-z| \geq \varepsilon$ (jump process) | jump from z to x $z \to x$

Transition probability per time unit $z \to x$

(ii) $\lim_{\Delta t\to 0} \frac{1}{\Delta t} \int_{|x-z|<\varepsilon} dx\,(x_i - z_i)\, p(x,t+\Delta t/\underbrace{z,t}_{\text{I.C.}}) = A_i(z,t) + O(\varepsilon)$ uniform in x,z,t

(continuous transitions)

(iii) $\lim_{\Delta t\to 0} \frac{1}{\Delta t} \int_{|x-z|<\varepsilon} dx\,(x_i-z_i)(x_j-z_j)\, p(x,t+\Delta t/z,t) = B_{ij}(z,t) + O(\varepsilon)$ uniform in z,ε,t

all higher-order moments vanish $O(\varepsilon)$!

- Consider $\frac{\partial}{\partial t} \int dx\, f(x)\, p(x,t/y,t')$ for any function $f(x)$

and derive from it a dif. eq. for $\frac{\partial}{\partial t} p(x,t/y,t')$:

$$\frac{\partial}{\partial t}\int dx\, f(x)\, p(x,t/y,t') = \lim_{\Delta t\to 0} \left\{ \int dx\, f(x)\, \frac{[\overbrace{p(x,t+\Delta t/y,t')} - p(x,t/y,t')]}{\Delta t} \right\} =$$

$$= \lim_{\Delta t \to 0} \left\{ \int dx\, f(x)\, \frac{\overbrace{\int dz\, p(x,t+\Delta t/z,t)\, p(z,t/y,t')}^{\text{use Chapman-Kolmogorov}} - p(x,t/y,t')}{\Delta t} \right\}$$

rename $x \to z$

for $|x-z| < \varepsilon$

Taylor series

$$f(x) = f(z) + \sum_i \frac{\partial f(z)}{\partial z_i}(x_i - z_i) + \sum_{ij} \frac{1}{2} \frac{\partial^2 f(z)}{\partial z_i \partial z_j}(x_i-z_i)(x_j-z_j) + \text{rest } (\to 0 \text{ for } |x-z| \to 0)$$

Split integrals $\int_{|x-z|<\varepsilon}$ and $\int_{|x-z|>\varepsilon}$

$$\frac{\partial}{\partial t}\int dx\, f(x)\, p(x,t/y,t')$$

$$= \lim_{\Delta t\to 0} \frac{1}{\Delta t} \left\{ \iint_{|x-z|<\varepsilon} dx\, dz\, \overbrace{\left[\underbrace{\sum_i (x_i - z_i) \frac{\partial f}{\partial z_i}}_{\text{(ii) } \boxed{A_i}} + \underbrace{\sum_{ij} \frac{1}{2}(x_i-z_i)(x_j-z_j) \frac{\partial^2 f}{\partial z_i \partial z_j}}_{\text{(iii) } \boxed{B_{ij}}} \right]}^{f} p(x,t+\Delta t/z,t)\, p(z,t/y,t') + \right.$$ (1)

$$+ \iint_{|x-z|<\varepsilon} \underbrace{\text{higher order terms}}_{\to 0} + \iint_{|x-z|<\varepsilon} dx\, dz\, f(z)\, \underbrace{p(x,t+\Delta t/z,t)}_{\to 0}\, p(z,t/y,t') +$$

$$+ \iint\limits_{|x-z|\geq\varepsilon} dx\,dz\, f(x)\, p(x,t+\Delta t/z,t)\, p(z,t/y,t') \quad - \quad ②$$

$$- \iint dx\,dz\, f(z)\, p(x,t+\Delta t/z,t)\, p(z,t/y,t') \Big\} \quad ③$$

(i) substitute $\boxed{W}$

$\int dx\, p(x,t+\Delta t/\ldots) = 1$ inserted

term ② $\varepsilon \to 0$ $\lim\limits_{\varepsilon\to 0} \int\limits_{|x-z|\geq\varepsilon} dx$ = principal value integral (assumption: exists)

for term ① partial integration (integr. by parts)

$$\int dz \left(\frac{\partial}{\partial z_i} f(z)\right) A_i(z)\, p(z,t/\ldots) = -\int dz\, f(z) \frac{\partial}{\partial z_i}\left(A_i\, p(z,t/\ldots)\right) + \text{terms without integral } (\to 0)$$

$$\int dz \left(\frac{\partial^2}{\partial z_i \partial z_j} f(z)\right) B_{ij}(z)\, p(z,t/\ldots) = +\int dz\, f(z) \frac{\partial^2}{\partial z_i \partial z_j}\left[B_{ij}(z)\, p(z,t/\ldots)\right]$$

$$\Rightarrow \int dz\, f(z) \left[\ldots\right] = 0 \quad \text{for any } f(z)$$

$$\left[\ldots\right] = 0$$

$$\frac{\partial}{\partial t} p(z,t/y,t') = -\sum_i \frac{\partial}{\partial z_i}\left[A_i(z,t)\, p(z,t/y,t')\right] + \quad ①$$

$$+ \sum_{ij} \frac{1}{2} \frac{\partial^2}{\partial z_i \partial z_j}\left[B_{ij}(z,t)\, p(z,t/y,t')\right] +$$

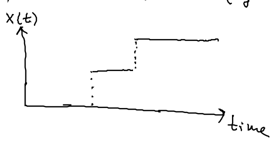
prob-ty of transition per unit time

$$+ \int dx \left[\underset{②}{W(z/x,t)}\, p(x,t/y,t') - \underset{③}{W(x/z,t)}\, p(z,t/y,t')\right]$$

<u>differential</u> <u>Chapman-Kolmogorov</u> <u>equation</u>

Initial conditions $p(z,t'/y,t') = \delta(y-z)$ $\quad t'$ = initial time

(a) jump process



$$\frac{\partial}{\partial t} p(z,t/\ldots) = \int dx \left[W(z/x,t)\, p(x,t/\ldots) - W(x/z,t)\, p(z,t/\ldots)\right]$$