4.3 Synchronization in the presence of noise. Effective synchronization.

Deterministic case

$$|m\varphi_1(t) - n\varphi_2(t)| = \text{const},$$

m, n are integers ; φ_1, φ_2 are phases of oscillators

$\Rightarrow$ <u>phase</u> locking

Sync can be also defined as <u>frequency</u> locking $\Rightarrow$

$\Rightarrow$ freq. of the oscillator and the freq. of the driving force (or another oscillator) are in rational relation : $1:1$ is the simplest case.

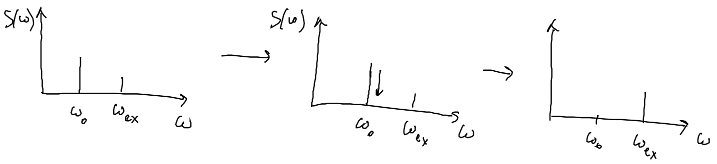
The ratio of the driving frequency to the mean frequency of the oscillator:

$$\theta = \frac{\Omega}{\langle\omega\rangle} \quad ; \quad \theta = 1 \Rightarrow 1:1$$

<u>Mechanisms</u> : ① phase / frequency locking

② suppression of intrinsic frequency

①


②


<u>What is the role of noise?</u>

The theory shows that noise counteracts sync: sync occurs only for a limited time interval. On the other hand $\rightarrow$ noise induces new regimes and switchings.

- signals are not periodic
- power spectrum is not discrete (may not contain peaks at any distinct frequencies)

Nevertheless, the concept of phase and freq. locking can be applied. The definitions from determ. theory can not be used directly, since the signals are not harmonic.

How to define the phase for a noisy system?

The introduction of the phase in a noisy oscillating system requires a probabilistic approach:

* instan. amplitude and phase are stochastic variables since $x(t)$, $\dot{x}(t)$ are stochastic

* SDE including a noise term $\xi(t)$

To extract inform. from stoch. dynamics we have to calculate moments of $A(t)$, $\varphi(t)$ and $\omega(t) = \dot{\varphi}(t)$

or consider transition prob. density

$p(A, \varphi, t \,/\, A^*, \varphi^*, t_0)$ which is sufficient for Markovian approximations.

$\swarrow$

conditional prob. to observe the ampl A and phase φ at time t if started at time t_0 with A^* and φ^*.

The stochastic process $\varphi(t)$ can be decomposed into two parts

a determ. part given by its mean value (or mean value of the instan. frequency)

a fluctuating part characterized, for example, by diffusion coefficient.

Synchronization $\longrightarrow$ fixed relation between two phases is always interrupted by randomly occurring abrupt changes in the phase difference, also known as phase slips.

$\Downarrow$

In a noisy system the notion of sync must be math-ly expressed by relations and conditions between the moments of the fluctuating phase or its corresp. probability density.

SDE : van der Pol oscillator + per. force + noise

$$\ddot{x} - (\varepsilon - x^2)\dot{x} + \omega_0^2 x = b \sin(\omega t) + \sqrt{2D_0}\,\xi(t)$$

$$\downarrow$$

$$\dot{\rho} = \frac{\varepsilon}{2}\rho - \frac{1}{8}\rho^3 - \beta\cos\varphi + \frac{D}{\rho} + \sqrt{2D}\,\xi_1(t)$$

$$\dot{\varphi} = -\Delta + \frac{\beta}{\rho}\sin\varphi + \frac{\sqrt{2D}}{\rho}\,\xi_2(t)$$

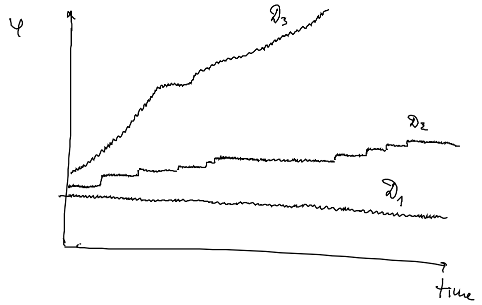
$$D = \frac{D_0}{(2\omega^2)}$$

$$\downarrow$$

$$\boxed{\dot{\varphi} = -\Delta + \frac{\beta}{\rho_0}\sin\varphi + \frac{\sqrt{2D}}{\rho_0}\,\xi_2(t)}$$

The dynamics of phase difference φ can be viewed as the motion of an overdamped Brownian particle in the tilted potential $U(\varphi)$ with the slope defined by Δ. The parameter $\frac{\beta}{\beta_0} = \Delta_s$ gives the hight of the potential barriers.

Noise $\Rightarrow$ diffusion of the phase difference in the potential $\Rightarrow$ $\varphi(t)$ fluctuates for a long time inside a potential well (which means phase locking) and rarely makes jumps from one well to another (displays phase slips) changing its value by 2π.



$\mathcal{D}_1 < \mathcal{D}_2 < \mathcal{D}_3$

We integrate the SDE numerically for different $\mathcal{D}$ values. For $\mathcal{D}_1$ (small noise) $\rightarrow$ phase difference remains bounded during long observ. time.

The increase of noise intensity leads to decrease of the average duration of residence times inside a potent. well and causes the hopping dynamics of the phase difference. For a large slope (detuning) and for a small value of periodic force amplitude, the jumps one metastable state to another become very frequent.

Phase description

mean angular velocity $\langle \omega \rangle$ and effective diffusion coef. D_{eff}

$$\langle \omega \rangle = \lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} \frac{d\varphi(t)}{dt}\, dt = \lim_{T \to \infty} \frac{1}{T} \left(\varphi(t_0 + T) - \varphi(t_0) \right)$$

$$D_{eff} = \frac{1}{2} \frac{d}{dt} \left[\langle \varphi^2(t) \rangle - \langle \varphi(t) \rangle^2 \right]$$

phase in the analytic signal representation

$x(t) \longrightarrow$ we construct an analytic signal $w(t)$ in complex plane

$$w(t) = x(t) + i\, y(t) = A(t)\, e^{i\varphi(t)}$$

$$A(t) = \sqrt{x^2(t) + y^2(t)}$$

$$\varphi(t) = \arctan\left(\frac{y}{x}\right) + \pi k, \quad k = 0; \pm 1; \pm 2, \dots$$

$$\omega(t) = \frac{d\varphi(t)}{dt} = \frac{1}{A^2(t)} \{ x(t)\dot{y}(t) - y(t)\dot{x}(t) \}$$

How to define $y(t)$?

Often Hilbert transform is used:

$$y(t) = H[x] = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x(\tau)}{t-\tau} d\tau =$$

$$= \frac{1}{\pi} \int_{0}^{\infty} \frac{x(t-\tau) - x(t+\tau)}{\tau} d\tau$$

$x(t) \to H\{x(t)\}$

$H[H\{x(t)\}] = -x(t)$

Properties: as a linear transformation $H\{x\}$ obeys several useful properties

* every Hilbert transform of a linear superposition of two signals is the superposition of the separate Hilbert transforms.

* time shift of the signal $\Rightarrow$ shift of the argument of the Hilbert transform

* the Hilbert transform of a Hilbert transform gives the negative original signal

* even functions give odd Hilbert transforms and vice versa.

* the original signal $x(t)$ and the Hilbert transform $H[x(t)]$ are orthogonal

* the full energy of the original signal, the integral of $x^2(t)$ over all times is equal to the energy of the transformed one.

There are also other ways to introduce the phase.

Phase sync in the presence of noise

$$\lim_{t\to\infty} |n\varphi_1(t) - m\varphi_2(t)| < \text{const}$$

θ is rotation number (= "winding number")

$\theta = m:n \to$ holds true in some finite region of system's parameter space $\to$ sync region

* frequency locking: means a rational ratio of two initially independent frequencies $\frac{\omega_1}{\omega_2} = \frac{m}{n}$

* phase locking: $\dot{\varphi} = 0$, $\varphi = \text{const}$

The influence of noise leads to destruction of the sync regime in the sense of the above given definition. However if the noise is small we can define effective sync.

Phase diffusion $\rightarrow$ the definition of the sync in the presence of noise appears to be "blurred" $\Rightarrow$
the sync conditions should be defined in a statistical way.