Cumulants and Moments

ϑ-th moment $M_\vartheta = \langle x^\vartheta \rangle$

Moment-generating function $Z(\alpha) = \langle e^{\alpha x} \rangle =$

$$= \sum_{\vartheta=0}^{\infty} \frac{\alpha^\vartheta}{\vartheta!} M_\vartheta \quad ; \quad \frac{\partial^\vartheta}{\partial \alpha^\vartheta} Z(\alpha)\Big|_{\alpha=0} = M_\vartheta$$

The moment-generating function is named so because it can be used to find the moments of the distribution

The knowledge about all moments is equivalent to knowing the probability density function.

$\alpha = is$: inverse Fourier transform

$$Z(is) = \int dx\, \rho(x)\, e^{isx}$$

$$\rho(x) = \frac{1}{\pi} \int ds\, Z(is)\, e^{-ixs}$$

Generalization to d random variables
(d dimensions)

$$M_{\vartheta_1 \vartheta_2 \dots \vartheta_d} = \langle x_1^{\vartheta_1} x_2^{\vartheta_2} \dots x_d^{\vartheta_d} \rangle$$

Moments of order $\vartheta = \vartheta_1 + \vartheta_2 + \vartheta_3 + \dots \vartheta_d$

Moment generating function

$$Z(\underline{\alpha}) = \langle e^{\underline{\alpha}\,\underline{x}} \rangle = \sum_{\vartheta_1 \dots \vartheta_d} \frac{\alpha_1^{\vartheta_1} \dots \alpha_d^{\vartheta_d}}{\vartheta_1! \dots \vartheta_d!} M_{\vartheta_1 \dots \vartheta_d}$$

$$\underline{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_d)$$

$$\underline{x} = (x_1, x_2, \dots, x_d)$$

Cumulant

Cumulant $C_{\nu_1 \dots \nu_d} = \langle x_1^{\nu_1} x_2^{\nu_2} \dots x_d^{\nu_d} \rangle_c$ defined by cumulant-generating function

$$\Gamma(\underline{\alpha}) = \ln \langle e^{\underline{\alpha}\,\underline{x}} \rangle = \sum_{\nu_1 \dots \nu_d} \frac{\alpha_1^{\nu_1} \dots \alpha_d^{\nu_d}}{\nu_1! \dots \nu_d!} C_{\nu_1 \dots \nu_d}$$

Properties: cumulants additive for uncorrelated random variables (not valid for moments!)

$$\langle (x_1 + x_2)^2 \rangle = \langle x_1^2 \rangle + \langle x_2^2 \rangle + 2 \langle x_1 x_2 \rangle$$

Proof: Let x_1, x_2 be uncorrelated $\underline{\alpha} = (\alpha_1, \alpha_2)$, $\underline{x} = (x_1, x_2)$

$$\Rightarrow Z(\alpha) = \langle e^{\underline{\alpha}\,\underline{x}} \rangle = \int dx_1\, dx_2\, \rho(x_1) \rho(x_2)\, e^{\alpha_1 x_1} e^{\alpha_2 x_2} =$$

$$= \langle e^{\alpha_1 x_1} \rangle \langle e^{\alpha_2 x_2} \rangle$$

$$\Rightarrow \Gamma(\underline{\alpha}) = \ln Z(\underline{\alpha}) = \ln \langle e^{\alpha_1 x_1} \rangle + \ln \langle e^{\alpha_2 x_2} \rangle \overset{!}{=} \Gamma(\alpha_1) + \Gamma(\alpha_2)$$

$$\alpha_1 = \alpha_2 = \alpha : \Gamma(\alpha, \alpha) = \ln \langle e^{\alpha \overbrace{(x_1 + x_2)}^{x}} \rangle =$$

$$= \sum_\nu \frac{\alpha^\nu}{\nu!} \langle (x_1 + x_2)^\nu \rangle_c =$$

$$= \sum_\nu \frac{\alpha^\nu}{\nu!} \langle x_1^\nu \rangle_c + \sum_\nu \frac{\alpha^\nu}{\nu!} \langle x_2^\nu \rangle_c$$

$$\Rightarrow \langle (x_1 + x_2)^\nu \rangle_c = \langle x_1^\nu \rangle_c + \langle x_2^\nu \rangle_c$$

$\square$