Lecture 16 summary

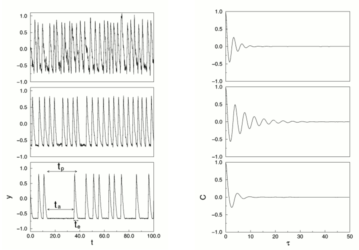
3.3 Coherence resonance

CR: the best temporal regularity (coherence) of noise-induced oscillations is achieved for intermediate / optimal noise intensity D_{opt}.

FHN system in excitable regime

$$\boxed{\begin{aligned} \varepsilon \dot{x} &= x - \frac{x^3}{3} - y \\ \dot{y} &= x + a + \sqrt{2D}\,\xi(t) \end{aligned}}$$

Time series ACF



D is too large

D is optimal

D is too small

Pikovsky & Kurths, 1997

Measures of CR

- normalized standard deviation of ISI

$$R_T = \frac{\sqrt{\langle t_{ISI}^2 \rangle - \langle t_{ISI} \rangle^2}}{\langle t_{ISI} \rangle}$$

- correlation time

$$t_{cor} = \frac{1}{\Psi(0)} \int_0^\infty |\Psi(s)|\, ds$$

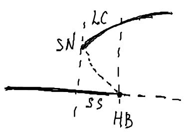
- signal-to-noise ratio

$$\beta = \frac{H}{\Delta\omega / \omega_p}$$

Hopf bifurcation

supercritical

Subcritical



3.4. Coherence resonance in non-excitable systems

CR in excitable systems

- excitability type I (SNIPER) - Haken 1993
- excitability type II (HB) FHN model - Pikovsky and Kurths 1997

CR in non-excitable systems

- Stuart-Landau oscillator - Ushakov et al. 2005 (+ semiconductor laser with delayed optical feedback)
- Duffing-van der Pol oscillator - Zakharova et al. 2010 (+ synthetic gene oscillator)

3.4.1 Hopf bifurcations

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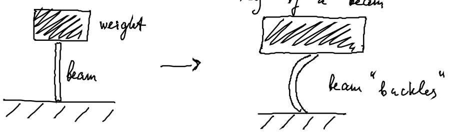
Deterministic Bifurcations

Bifurcation – qualitative change of the phase portrait which occurs when one or several control parameters are tuned

- structural changes of phase portrait
- appearance / disap. of limit sets
- changes in stability of trajectories

A simple example: buckling of a beam



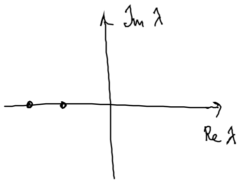
weight – control parameter
deflection of the beam – dynamical variable x

Hopf Bifurcation

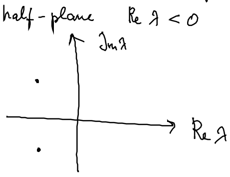
2D system with a stable fixed point.
What are all the possible ways it could lose stability? (as parameter μ varies)

$$\dot{\vec{x}} = \vec{F}(\vec{x}, \vec{\mu})$$

fixed point is stable $\Rightarrow$ λ_1, λ_2 must both lie in the left half-plane $\mathrm{Re}\,\lambda < 0$



λ_1, λ_2 are real and negative



λ_1, λ_2 are complex conjugates

To destabilize the fixed point $\rightarrow$ λ_1, λ_2 cross into the right half-plane

Supercritical HB

Example
$$\boxed{\begin{aligned} \dot{r} &= \mu r - r^3 \\ \dot{\theta} &= \omega + b r^2 \end{aligned}}$$

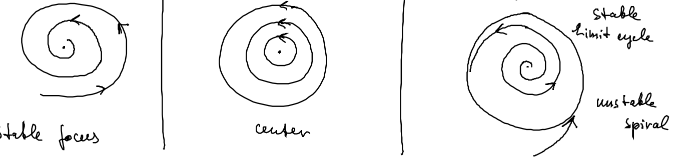
μ – control parameter
ω – frequency of oscillation
b – the dependence of frequency on amplitude for large-amplitude oscillations

Stable focus (spiral) changes into unstable spiral surrounded by a small LC.

HB occurs in phase spaces of any dimension $n \geq 2$

Phase space

$\mu < 0$ (before HB)	$\mu = 0$	$\mu > 0$ (after HB)
$\mathrm{Re}\,\lambda_{1,2} < 0$	$\lambda_{1,2} = \pm i\omega$	$\mathrm{Re}\,\lambda_{1,2} > 0$



stable focus | center

$x = r\cos\theta,\ y = r\sin\theta$ — change variables

$\Rightarrow \dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta = (\mu r - r^3)\cos\theta -$
$- r\,[\omega + b r^2]\sin\theta = (\mu - [x^2+y^2])x -$
$- (\omega + b[x^2+y^2])y = \mu x - \omega y +$ cubic terms

$\dot{y} = \omega x + \mu y +$ cubic terms

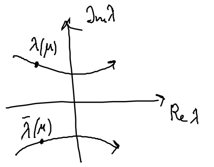
The Jacobian at the origin

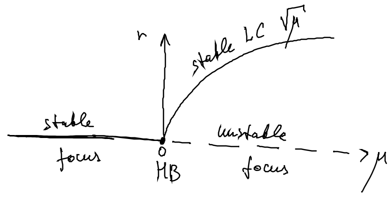
$$A = \begin{pmatrix} x & -\omega \\ \omega & \mu \end{pmatrix}$$

$\lambda_{1,2} = \mu \pm i\omega \quad \Rightarrow$ The eigenvalues cross the imaginary axis from left to right as μ increases from negative to positive values.

Assumptions / idealized example

- in HB in practice, the LC can be elliptical, not circular $\Rightarrow$ our example only typical topologically, not geometrically
- eigenvalues $\to$ on horizontal lines as μ varies

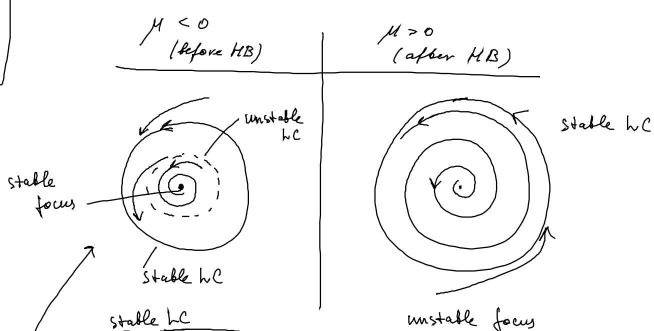


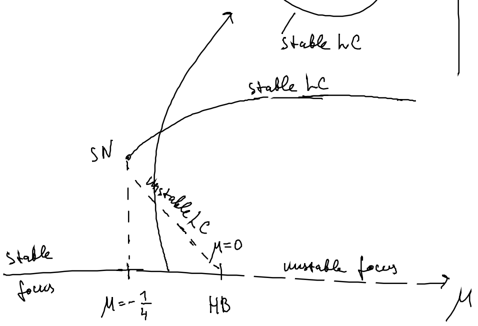


subcritical HB

Phase space

$$\boxed{\begin{aligned} \dot{r} &= \mu + r^3 - r^5 \\ \dot{\theta} &= \omega + b r^2 \end{aligned}}$$





hysteresis: once large-amplitude oscillations have begun, they cannot be turned off by bringing μ back to zero $\to$ large-amplitude osillations will persist.

Applications: subcritical HB is always much more dramatic and potentially dangerous in engineering applications

* aeroelastic flutter and other vibrations of airplane wings (Dowell & Ilgatova 1988, Thompson & Stewart 1986)

* instabilities of fluid flows (Drazin & Reid 1981)

* dynamics of nerve cells (Rinzel & Ermentrout 1989)

3.4.2 Stochastic Bifurcations

Bifurcations

deterministic

$$\dot{\vec{x}} = \vec{F}(\vec{x}, \vec{\mu})$$

parameters $\vec{\mu}$

stochastic

$$\dot{\vec{x}} = \vec{F}(\vec{x}, \vec{\mu}) + \xi(t)$$

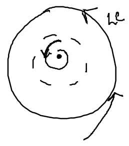
noise

noise intensity is the parameter

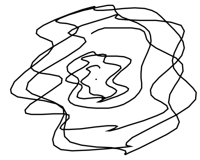
What do we have in the phase space?

Attractors

bistability: two attractors



One invariant set of trajetories

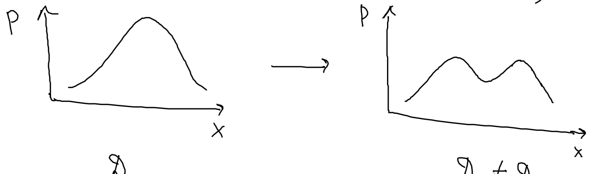


- dependence on IC disappears
- one can not distinguish between attractors

How to characterize the dynamics of stochastic systems?
How to track the dynamical changes in the stoch. case?

Stochastic Bifurcations

- Phenomenological (P-bifurcations) stochastic bifurcation
  a qualitative change of the stationary probability distribution.
  (transition from a unimodal distribution to a bimodal → → change of the number of maxima)



$\mathcal{D}_1$ $\qquad$ $\mathcal{D}_2 \neq \mathcal{D}_1$

L. Arnold, Random Dynamical System, Springer, Berlin 2003