Lecture 23 summary

4 Sync in the presence of noise

4.1 What is sync? Classical example of a pendulum clock.

4.2 Sync of periodic self-sustained oscillations

Forced sync of van der Pol oscillator:

truncated equations for amplitude and phase

$$\ddot{x} - (\varepsilon - x^2)\dot{x} + \omega_0^2 x = b \sin(\omega t) \quad (4.1)$$

$$\ddot{x} + \omega^2 x = (\omega^2 - 1)x + (\varepsilon - x^2)\dot{x} + b\sin(\omega t) \quad (4.2)$$

Assumptions

- $\omega \sim 1$ $(\omega_0 = 1)$
- the control parameter ε is small and positiv
- the external force is weak (its amplitude is small)

The r.h.s of (4.2) represents a weak perturbation of the harmonic oscillator with frequency ω, and the nonautonomous oscillator is a quasilinear or weakly linear system.

In this case a solution of (4.2) can be represented in the form:

$$x(t) = \mathrm{Re}\,\{a(t)\exp(i\omega t)\} = \frac{1}{2}\{a\exp(i\omega t) + a^*\exp(-i\omega t)\} \quad (4.3)$$

with additional condition

$$\dot{a}\exp(i\omega t) + \dot{a}^*\exp(-i\omega t) = 0 \quad (4.4)$$

The function $a(t)$ is assumed to be slowly varying compared with the period $T = \frac{2\pi}{\omega}$. (4.4) $\Rightarrow$ we can find first and second derivatives:

$$\dot{x} = \frac{1}{2}\{\dot{a}\exp(i\omega t) + \dot{a}^*\exp(-i\omega t) + i\omega a\exp(i\omega t) - i\omega a^*\exp(-i\omega t)\} =$$

$$= \frac{1}{2}[i\omega a\exp(i\omega t) - i\omega a^*\exp(-i\omega t)],$$

$$\ddot{x} = \frac{1}{2}\{i\omega\dot{a}\exp(i\omega t) - i\omega\dot{a}^*\exp(-i\omega t) -$$

$$- \omega^2 a\exp(i\omega t) - \omega^2 a^*\exp(-i\omega t)\} =$$

$$= i\omega\dot{a}\exp(i\omega t) - \frac{\omega^2}{2}\{a\exp(i\omega t) + a^*\exp(-i\omega t)\}$$

Substituting $\ddot{x}$, $\dot{x}$ and x into (4.2) and expressing $\sin(\omega t)$ in terms of exp. functions, we obtain

$$i\omega \dot{a} \exp(i\omega t) = \frac{\omega^2 - 1}{2} [a \exp(i\omega t) +$$
$$+ a^* \exp(-i\omega t)] + \frac{b}{2i} [\exp(i\omega t) - \exp(-i\omega t)] +$$
$$+ \frac{i\omega}{2} \{ \varepsilon - \frac{1}{4} [a^2 \exp(2i\omega t) + 2|a|^2 +$$
$$+ (a^*)^2 \exp(-2i\omega t)] \} [a \exp(i\omega t) - a^* \exp(-i\omega t)],$$
$$\dot{a} = \frac{\omega^2 - 1}{2i\omega} [a + a^* \exp(-2i\omega t)] - \frac{b}{2\omega} [1 - \exp(-2i\omega t)] +$$
$$+ \left\{ \frac{\varepsilon}{2} - \frac{1}{8} [a^2 \exp(2i\omega t) + 2|a|^2 + (a^*)^2 \exp(-2i\omega t)] \right\} [a -$$
$$- a^* \exp(-2i\omega t)]$$

Averaging over period $T = \frac{2\pi}{\omega}$ on r.h.s and l.h.s. of the equation, then taking into account the fact that $a(t)$ is a slowly varying function, we obtain the truncated equation for the complex amplitude in the form:

$$\dot{a} = -i \frac{\omega^2 - 1}{2\omega} a + \frac{\varepsilon}{2} a - \frac{1}{8} |a|^2 a - \frac{b}{2\omega} \quad (4.5)$$

Representing the complex quantity $a(t)$ in polar coordinates

$$a(t) = \rho(t) \exp(i\varphi(t)), \quad (4.6)$$

we derive the truncated equations

$$\dot{\rho} = \frac{\varepsilon}{2}\rho - \frac{1}{8}\rho^3 - \beta \cos\varphi \quad (4.7)$$

$$\dot{\varphi} = -\Delta + \frac{\beta}{\rho} \sin\varphi \quad (4.8)$$

$\Delta = (\omega^2 - 1)/2\omega \rightarrow$ detuning between the frequency of the external force and the natural frequency of the oscillator.

$\beta = \frac{b}{2\omega} \rightarrow$ intensity (or amplitude) of the external force

$\varphi \rightarrow$ phase $\rightarrow$ difference between phases of the oscillator and the force.

Sync $\rightarrow$ adjustment of the oscillator frequency to the frequency of the external force. In this case the force significantly influences the phase, but has only small effect on the amplitude $\Rightarrow$ the sync can be described in the phase approximation.

Analysis of sync in the phase approximation

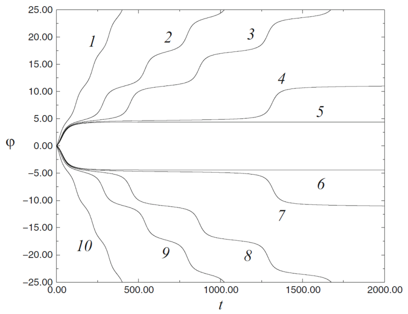
external force is weak $\Rightarrow$ one can assume that the amplitude $\rho(t)$ corresponds to the radius of the L.C. of the autonomous system

$\rho(t) = 2\sqrt{\varepsilon}$

$\Rightarrow (4.8) \longrightarrow \quad \frac{d\varphi}{dt} = -\Delta + \frac{\beta}{2\sqrt{\varepsilon}} \sin\varphi \quad (4.9)$

The equation (4.9) describes one dynamical variable φ, so the phase space dimension is 1. The system depends on parameters: Δ, β, ε.

Let us consider the phase dynamics in relation to the detuning Δ and the external amplitude β for a fixed value of $\varepsilon = 0.1$, $\beta = 0.01$.



Temporal phase dynamics $\varphi(t)$ for $\varepsilon = 0.1$, $\beta = 0.01$ and different values of detuning: $\Delta = \mp 0.07$ for lines 1 and 10; $\Delta = \mp 0.04$ for lines 2 and 9; $\Delta = \mp 0.035$ for lines 3 and 8; $\Delta = \mp 0.032$ for lines 4 and 7; and $\Delta = \mp 0.03$ for lines 5 and 6.

For small detuning the phase is const (lines 5 and 6).
As the frequency detuning grows and when $|\Delta|$ exceed a certain critical value $|\Delta_c|$, the phase beh. $\varphi(t)$ changes qual-ly. In fact, its value starts varying in time.

According to the sign of Δ (if the freq. of the forcing is greater or less than the natural freq.), the phase φ either decreases or increases in time.
For small subcriticality $|\Delta - \Delta_c|$, the time series $\varphi(t)$ shows long intervals during which the phase is const. (nearly)
These long intervals intermingle with relatively short time intervals where the phase changes by 2π. As the supercriticality grows, the intervals of constant phase decrease, and the mean rate of phase change increases.
Thus, there is an interval of values of the deduning $|\Delta| < |\Delta_c|$ where the phase is constant, $\varphi(t) = \text{const}$, and its derivative (the rate of phase change) is zero. This means the system oscillate periodically at the frequency of the ext. force $\Rightarrow$ sync!

Outside the sync region the phase changes in time and the oscillations become quasiperiodic. The mean rate of the phase change $\langle \dot\varphi(t) \rangle$ defines the second frequency, the so-called beat frequency.

In the phase approx., sync motions correspond to fixed points of dynamical system (4.9):

$$\frac{d\varphi}{dt} = -\Delta + \frac{\beta}{2\sqrt{\varepsilon}} \sin\varphi \qquad (4.9)$$

Regimes of sync are related to stable fixed points.

$$-\Delta + \frac{\beta}{2\sqrt{\varepsilon}} \sin\varphi = 0 \qquad (4.10)$$

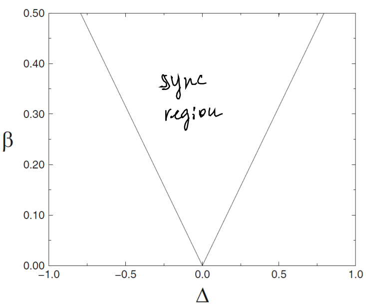
We find fixed points:

$$\varphi_1 = \arcsin \frac{2\Delta\sqrt{\varepsilon}}{\beta} \qquad (4.11)$$

$$\varphi_2 = \pi - \arcsin \frac{2\Delta\sqrt{\varepsilon}}{\beta} \qquad (4.12)$$

They exist if $|2\Delta\sqrt{\varepsilon}/\beta| \leq 1$ or $|\Delta| \leq \frac{\beta}{2\sqrt{\varepsilon}}$. If we fix ε,

the region of existence of the fixed points in the parameter plane (β, Δ) is bounded by the line given by the equation: $\beta = 2\sqrt{\varepsilon}\,|\Delta|$ (4.13)



Sync region in the plane of the parameters Δ and β for $\varepsilon = 0.1$

Let us analyze the stability of the fixed points. The beh. of system (4.9) is considered in the neighbourhood of fixed points φ_i $(i = 1, 2)$ in a linear approx. We represent the dynamical variable $\varphi(t)$ in the form:

$\varphi(t) = \varphi_i + \tilde{\varphi}(t)$, where $\tilde{\varphi}(t)$ is a small deviation from the fixed point φ_i.

We can rewrite (4.9) as follows:

$$\frac{d}{dt}(\varphi_i + \tilde{\varphi}) = -\Delta + \frac{\beta}{2\sqrt{\varepsilon}} \sin(\varphi_i + \tilde{\varphi}),$$

$$\frac{d(\varphi_i)}{dt} = 0 \quad \text{(in the fixed point)}$$

$$\frac{d\tilde{\varphi}}{dt} = -\Delta + \frac{\beta}{2\sqrt{\varepsilon}} (\sin\varphi_i \cos\tilde{\varphi} + \sin\tilde{\varphi} \cos\varphi_i)$$

Expanding $\cos\tilde{\varphi}$ and $\sin\tilde{\varphi}$ in Taylor series and taking the first-order terms in $\tilde{\varphi}$, we obtain the linearized equation:

$$\frac{d\tilde{\varphi}}{dt} = -\Delta + \frac{\beta}{2\sqrt{\varepsilon}} \left(\sin\varphi_i + \tilde{\varphi} \cos\varphi_i \right)$$

From the definition of the fixed point:

$$-\Delta + \frac{\beta}{2\sqrt{\varepsilon}} \sin\varphi_i = 0$$

$$\Rightarrow \frac{d\tilde{\varphi}}{dt} = \left(\frac{\beta}{2\sqrt{\varepsilon}} \cos\varphi_i \right) \tilde{\varphi} \qquad (4.14)$$

solution $\tilde{\varphi}(t) \sim \exp\left\{ \left(\frac{\beta}{2\sqrt{\varepsilon}} \cos\varphi_i \right) t \right\}$ (4.15)

The stability of fixed points depends on the sign of $\cos\varphi_i$. If $\cos\varphi_i < 0$, then small deviation decays in time $\Rightarrow$ the fixed point is stable.
If $\cos\varphi_i > 0 \Rightarrow$ the fixed point is unstable.