

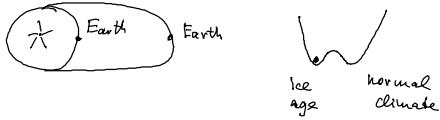
lecture 12 summary

3. Noise-induced oscillations and patterns

3.1 Stochastic resonance (SR)

SR is a mechanism by which a nonlinear dynamical system in a noisy environment gets an enhanced sensitivity towards small external periodic force for an interm./optimal noise intensity.

Example 1 Periodicity of ice age on the Earth



Example 2 overdamped Brownian particle in a bistable potential

$$V(x) = -\frac{a}{2}x^2 + \frac{b}{4}x^4$$

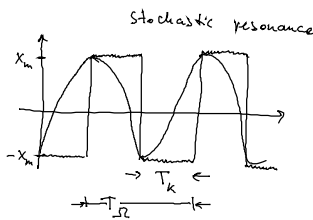
$$\dot{x} = -V'(x) + A_0 \cos(\Omega t) + g(t)$$

↑ noise
↑ periodic force

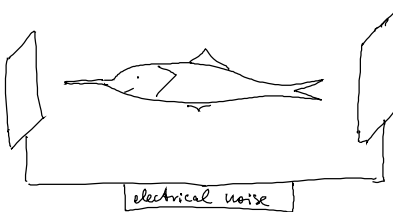
$$\langle x(t) \rangle = \bar{x} \cos(\Omega t - \bar{q}_0)$$

$$T_{\Omega} \approx 2T_k(\bar{\vartheta}) \quad \text{SR}$$

$$T_{\Omega} = \frac{2\pi}{\Omega}; \quad T_k \text{ - mean first passage time}$$

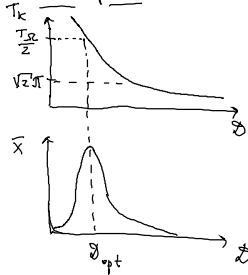


Example 3: Paddlefish



interm./optimal noise allows the fish to detect the largest amount of food.

Example 2



for small amplitudes

$$\bar{x}(\bar{\vartheta}) = \frac{A_0 \langle x^2 \rangle_0}{\bar{\vartheta}} \frac{2r_k(\bar{\vartheta})}{\sqrt{4r_k^2(\bar{\vartheta}) + \Omega^2}}$$

$$\bar{q}_0(\bar{\vartheta}) = \arctan \frac{\Omega}{2r_k} \quad \text{phase lag}$$

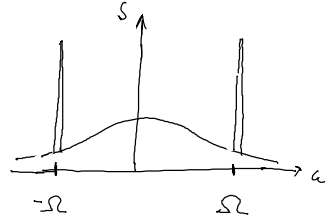
Power spectral density

$$S(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt e^{-i\omega t} \langle x(t+\tau)x(t) \rangle$$

Background noise $S_N(\omega) \approx \frac{4k_B \langle x^2 \rangle}{4k_B^2 + \omega^2}$

superimposed with δ peaks at $\omega = \pm \Omega$

$$S(\omega) = \frac{\pi}{2} \bar{x}(\theta)^2 [\delta(\omega - \Omega) + \delta(\omega + \Omega)] + S_N(\omega)$$



Signal-to-noise ratio: measure for signal enhancement

$$SNR = \frac{2 \lim_{\Delta\omega \rightarrow 0} \int_{-\Omega-\Delta\omega}^{-\Omega+\Delta\omega} S(\omega) d\omega}{S_N(\Omega)} \approx \pi \left(\frac{A_0 x_m}{\Omega} \right)^2 k_B(\theta) \sim \frac{e^{-\frac{\Delta V}{k_B T}}}{\Omega^2}$$

3.2 Noise-induced oscillations

Now we consider autonomous systems, i.e., systems without external periodic forcing.

Assumption \rightarrow determ. system has a stable fixed point.

\Rightarrow Noise can induce self-sustained oscillations
(stochastic limit cycle)

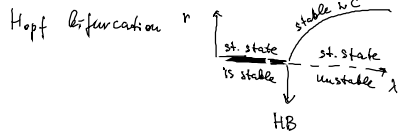
Reviews: Lindner, Garcia-Ojalvo, Neiman, Schimansky-Geier:

Effects of noise in excitable systems, Phys. Rep. 392, 321 (2004)

Janson, Balanov, Schöll: Control of noise-induced dynamics.

In: Handbook of chaos control (Wiley, 2008)

- often below the deterministic bifurcation related to occur. of limit cycle.



SNIPER = saddle-node infinite period

1. Example van der Pol oscillator

$$\begin{cases} \dot{x} = y \\ \dot{y} = (\varepsilon - x^2)y - \omega_0^2 x + \sqrt{2\theta} g(t) \end{cases}$$

θ - noise intensity
 $\theta = \sqrt{2\theta}$

$$\ddot{x} - \underbrace{(\varepsilon - x^2)}_{\text{SNIPER}} \dot{x} + \omega_0^2 x = \theta g(t)$$

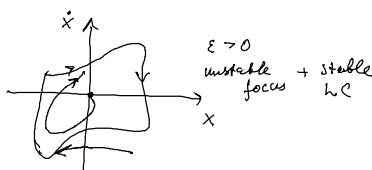
$$\ddot{x} + \mu(x^2 - 1)\dot{x} + \omega_0^2 x = 0 \quad \text{without noise}$$

1920 B. van der Pol

The model comes from nonlinear electrical circuits used in first radio devices.

Nonlinear damping term acts as positive damping for $|x| > 1$, but as negative damping for $|x| < 1$

\Rightarrow it causes large-amplitude oscillations to decay and it pumps them back up if they become too small.



$\mathcal{D} = 0$ (determin.): steady states $x^* = y^* = 0$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & \varepsilon \end{pmatrix}$$

$$\lambda^2 - \lambda \operatorname{tr} A + \det A = 0, \quad \operatorname{tr} A = \varepsilon, \quad \det = \omega_0^2 > 0$$

$$\Rightarrow A = \frac{\varepsilon}{2} \pm i \sqrt{\omega_0^2 - \left(\frac{\varepsilon}{2}\right)^2}$$

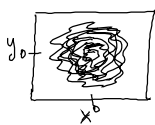
$\varepsilon = 0$ Hopf bifurcation ($\lambda = \pm i\omega_0$)

$\varepsilon < 0$ stable focus

$\varepsilon > 0$ unstable focus + LC

For example, $\varepsilon = -0.01$, $\omega_0 = 1$

\Rightarrow noise-induced oscillations ($\mathcal{D} \neq 0$)



$\mathcal{D} = 0.003$



$\mathcal{D} = 0.5$

Fig. 1 Pomplun et al.

Europhys. Lett,
71, 366 (2005)

2. Example Fitzhugh-Nagumo model (FHN)

excitable system (type II)

Application: spiking of neurons/neural populations

$$\begin{cases} \varepsilon \dot{x} = x - \frac{x^3}{3} - y \\ \dot{y} = x + a + \mathcal{D}g(t) \end{cases} \quad \text{FHN with noise}$$

$$\varepsilon \dot{u} = u - \frac{u^3}{3} - v \quad u - \text{activator (fast)}$$

$$\dot{v} = u + a + \sqrt{2\varepsilon}g(t) \quad v - \text{inhibitor (slow)}$$

This is an example of relaxation oscillator, typical for spike generations in a neuron after stimulation by an external input.

short nonlinear elevation of membrane voltage u , diminished over time by a slower, linear recovery variable v

ε - time scale separation ($\varepsilon = 0.01$)

$|a| > 1$ excitable regime ($a = 1.001$) ← bifurcation parameter

$\mathcal{D}=0$: steady states: $x = -a$
 $y = -a + \frac{a^3}{3}$

stab. $\begin{pmatrix} \delta \dot{x} \\ \delta \dot{y} \end{pmatrix} = \frac{1}{\varepsilon} \begin{pmatrix} 1-a^2 & -1 \\ \varepsilon & 0 \end{pmatrix}$, $\begin{matrix} \text{Jacobian matrix} \\ \text{tr } A = 1-a^2 \\ \text{det } A = \varepsilon > 0 \end{matrix}$

$a = 1$ Hopf bif.

$a < 1$ unstable st. state + LC (oscillatory regime)

$a > 1$ stable st. state (node) excitable regime

(for example: $a = 1.1, \varepsilon = 0.01$)

