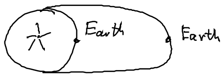
Lecture 12 summary

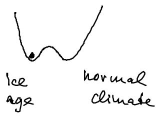
3. Noise-induced oscillations and patterns

3.1 Stochastic resonance (SR)

SR is a mechanism by which a nonlinear dynamical system in a noisy environment gets an enhanced sensitivity towards small external periodic force for an interm. / optimal noise intensity.

Example 1 Periodicity of ice age on the Earth





Example 2 overdamped Brownian particle in a bistable potential

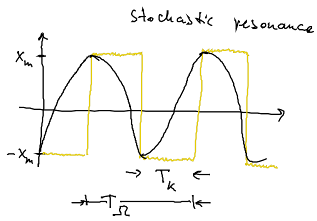
$$V(x) = -\frac{a}{2}x^2 + \frac{b}{4}x^4$$

$$\boxed{\dot{x} = -V'(x) + \underbrace{A_0 \cos(\Omega t)}_{\text{periodic force}} + \overbrace{\xi(t)}^{\text{noise}}}$$

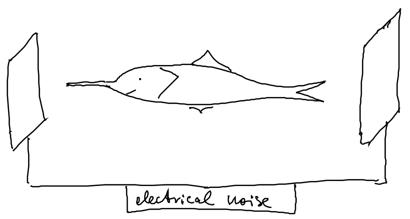
$$\langle x(t) \rangle = \bar{x} \cos(\Omega t - \bar{\varphi})$$

$$\boxed{T_\Omega \approx 2 T_K(D)} \quad SR$$

$T_\Omega = \frac{2\pi}{\Omega}$; T_K – mean first passage time

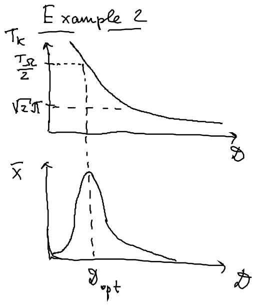


Example 3: Paddlefish



interm. / optimal noise allows the fish to detect the largest amount of food.

Example 2

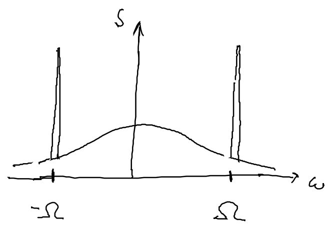


for small amplitudes

$$\bar{x}(D) = \frac{A_0 \langle x^2 \rangle_0}{D} \frac{2 r_K(D)}{\sqrt{4 r_K^2(D) + \Omega^2}}$$

$$\bar{\varphi}(D) = \arctan \frac{\Omega}{2 r_K} \quad \text{phase lag}$$

Power spectral density



$$S(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\tau \, e^{-i\omega\tau} \langle x(t+\tau)\, x(t) \rangle$$

Background noise $S_N(\omega) \approx \frac{4 r_k \langle x^2 \rangle_0}{4 r_k^2 + \omega^2}$

superimposed with δ peaks at $\omega = \pm \Omega$

$$S(\omega) = \frac{\pi}{2} \bar{x}(D)^2 \left[\delta(\omega - \Omega) + \delta(\omega + \Omega) \right] + S_N(\omega)$$

Signal-to-noise ratio: measure for signal enhancement

$$SNR = \frac{2 \lim_{\Delta\omega \to 0} \int_{\Omega - \Delta\omega}^{\Omega + \Delta\omega} S(\omega)\, d\omega}{S_N(\Omega)} \approx \pi \left(\frac{A_0 x_m}{D} \right)^2 r_k(D) \sim \frac{e^{-\frac{\Delta V}{D}}}{D^2}$$

3.2 Noise-induced oscillations

Now we consider autonomous systems, i.e., systems <u>without</u> external periodic forcing.

Assumption $\rightarrow$ determ. system has a stable fixed point.

$\Rightarrow$ Noise can induce self-sustained oscillations

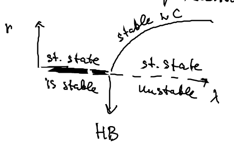
(stochastic limit cycle)

Reviews : Lindner, Garcia-Ojalvo, Neiman, Schimansky-Geier: Effects of noise in excitable systems, Phys. Rep. 392, 321 (2004)

Janson, Balanov, Schöll: Control of noise-induced dynamics. In: Handbook of chaos control (Wiley, 2008)

- often below the deterministic bifurcation related to occur. of limit cycle.

Hopf bifurcation



SNIPER = saddle-node infinite period

1. Example van der Pol oscillator

$$\checkmark \quad \boxed{\begin{aligned} \dot{x} &= y \\ \dot{y} &= \underbrace{(\varepsilon - x^2) y} - \omega_0^2 x + \sqrt{2D}\, \xi(t) \end{aligned}}$$

$\tilde{D}$ - noise intensity

$\tilde{D} = \sqrt{2D}$

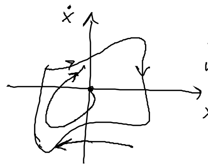
$$\ddot{x} - \underbrace{(\varepsilon - x^2)\dot{x}} + \omega_0^2 x = \tilde{D}\, \xi(t)$$

$$\boxed{\ddot{x} + \underline{\mu(x^2-1)\dot{x}} + \omega_0^2 x = 0}$$ without noise 1920 B. van der Pol

The model comes from nonlinear electrical circuits used in first radio devices.

Nonlinear damping term acts as positive damping for $|x|>1$, but as negative damping for $|x|<1$

$\Rightarrow$ It causes large-amplitude oscillations to decay and it pumps them back up if they become too small.



$\varepsilon > 0$ unstable focus + stable LC

$\mathcal{D} = 0$ (determ.): steady states $x^* = y^* = 0$

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = A\begin{pmatrix}x\\ y\end{pmatrix}, \quad A = \begin{pmatrix}0 & 1\\ -\omega_0^2 & \varepsilon\end{pmatrix}$$

$\lambda^2 - \lambda \,\mathrm{tr}A + \det A = 0, \quad \mathrm{tr}A = \varepsilon, \quad \det = \omega_0^2 > 0$

$$\Rightarrow \lambda = \frac{\varepsilon}{2} \pm i\sqrt{\omega_0^2 - \left(\frac{\varepsilon}{2}\right)^2}$$

$\varepsilon = 0$ Hopf bifurcation $(\lambda = \pm i\omega_0)$

$\varepsilon < 0$ stable focus

$\varepsilon > 0$ unstable focus + LC

For example, $\varepsilon = -0.01$, $\omega_0 = 1$

$\Rightarrow$ noise-induced oscillations $(\mathcal{D} \neq 0)$

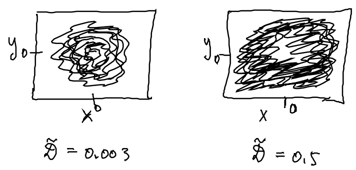


Fig. 1 Pomplun et al. Europhys. Lett. 71, 366 (2005)

2. Example FitzHugh–Nagumo model (FHN)

excitable system (type II)

<u>Application</u>: spiking of neurons / neural populations

$$\checkmark \quad \boxed{\begin{aligned} \varepsilon \dot{x} &= x - \frac{x^3}{3} - y \\ \dot{y} &= x + a + \mathcal{D}\xi(t) \end{aligned}}$$ FHN with noise

$\varepsilon \dot{u} = u - \frac{u^3}{3} - v$ $\qquad$ u – activator (fast)

$\dot{v} = u + a + \sqrt{2\mathcal{D}}\,\xi(t)$ $\qquad$ v – inhibitor (slow)

This is an example of relaxation oscillator, typical for <u>spike generations</u> in a neuron after stimulation by an external input.

Short nonlinear elevation of membrane voltage u, diminished over time by a slower, linear recovery variable v

ε – time scale separation $(\varepsilon = 0.01)$

$|a| > 1$ excitable regime ($a = 1.001$) $\leftarrow$ bifurcation parameter

$\underline{\mathcal{D} = 0}$: steady states : $x = -a$

$$y = -a + \frac{a^3}{3}$$

Jacobian matrix

stab. $\begin{pmatrix} \delta \dot{x} \\ \delta \dot{y} \end{pmatrix} = \frac{1}{\varepsilon} \begin{pmatrix} 1-a^2 & -1 \\ \varepsilon & 0 \end{pmatrix}$, $\operatorname{tr} A = 1 - a^2$

$\det A = \varepsilon > 0$

$a = 1$ Hopf bif.

$a < 1$ unstable st. state + LC (oscillatory regime)

$a > 1$ stable st. state (node) excitable regime

(for example : $a = 1.1$, $\varepsilon = 0.01$)