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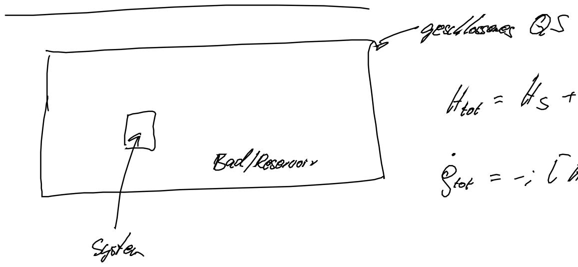
1. Offene Quantensysteme



$$H_{tot} = H_S + H_B + H_I$$

$$\dot\rho_{tot} = -i\left[H_{tot}, \rho_{tot}\right] \rightsquigarrow \rho_{tot}(t) = e^{-iH_{tot}\cdot t}\rho_{tot}(0)\, e^{+iH_{tot}\cdot t}$$

Suche: DM für das System ρ_S

Was ist die Entwicklungsgleichung für ρ_S?

$$\dot\rho_S \neq -i\left[H_{eff}(t), \rho_S\right] \rightsquigarrow \rho_S(t) \neq U_{eff}(t)\,\rho_S(0)\,U_{eff}^\dagger(t) \qquad U_{eff}^\dagger = U_{eff}^{-1}$$

1.1. Kraus Abbildung

$$\underset{t=0}{\rho} \to \underset{t>0}{\rho'} = \sum_{\alpha\beta} \underset{\in\mathbb{C}}{\gamma_{\alpha\beta}}\, \overset{\text{Operatoren}}{A_\alpha}\, \rho\, A_\beta^\dagger$$

$$\sum_{\alpha\beta}\gamma_{\alpha\beta}A_\beta^\dagger A_\alpha = \underline{\mathbb{1}}$$

$$\gamma_{\alpha\beta} = \left(\gamma_{\beta\alpha}\right)^* \qquad \left(\gamma = \gamma^\dagger\right)$$

$$\sum_{\alpha\beta} x_\alpha^* \gamma_{\alpha\beta} x_\beta \geq 0 \quad \forall x_\alpha \qquad (\underset{\text{EW von }\gamma}{\gamma_\alpha} \geq 0)$$

Vereinfachung $\quad A_\alpha = \sum_{\alpha'} \underset{\in\mathbb{C}}{U_{\alpha\alpha'}}\, \overset{\text{Operator}}{\bar K_{\alpha'}}$

$$\rho' = \sum_{\alpha\beta}\sum_{\alpha'\beta'} \gamma_{\alpha\beta}\, U_{\alpha\alpha'}\bar K_{\alpha'}\,\rho\, U_{\beta\beta'}^*\bar K_{\beta'}^\dagger$$

$$= \sum_{\alpha'\beta'} \bar K_{\alpha'}\,\rho\,\bar K_{\beta'}^\dagger \underbrace{\sum_{\alpha\beta} U_{\alpha\alpha'}\gamma_{\alpha\beta}U_{\beta\beta'}^*}_{\gamma_{\alpha'}\delta_{\alpha'\beta'}} = \sum_\alpha \underset{\geq 0}{\gamma_\alpha}\, \bar K_\alpha\,\rho\,\bar K_\alpha^\dagger$$

EW von $(\gamma_{\alpha\beta})$

$$K_\alpha = \sqrt{\gamma_\alpha}\cdot\bar K_\alpha$$

$$\boxed{\rho' = \sum_\alpha K_\alpha \rho K_\alpha^\dagger \qquad \sum_\alpha K_\alpha^\dagger K_\alpha = \mathbb{1}}$$

Kraus-Abbildung

erhält sämtliche DM-Eigenschaften

- $\rho = \rho^\dagger \rightsquigarrow (\rho')^\dagger = \sum_\alpha K_\alpha \rho^\dagger K_\alpha^\dagger = \sum_\alpha K_\alpha \rho K_\alpha^\dagger = \rho'$
- $\mathrm{Tr}\{\rho\} = 1 \rightsquigarrow \mathrm{Tr}\{\rho'\} = \sum_\alpha \mathrm{Tr}\{K_\alpha \rho K_\alpha^\dagger\} = \sum_\alpha \mathrm{Tr}\{K_\alpha^\dagger K_\alpha \rho\} = \mathrm{Tr}\{\underbrace{\sum_\alpha K_\alpha^\dagger K_\alpha}_{\mathbb{1}} \rho\} = 1$
- $\langle\psi|\rho|\psi\rangle \geq 0 \quad \forall |\psi\rangle$

$\rho = \sum_n \lambda_n |n\rangle\langle n|$ Spektralzerlegung

EW von ρ $\quad \langle n|m\rangle = \delta_{nm}$

$\lambda_n \geq 0$

$$\langle\psi|\rho'|\psi\rangle = \sum_\alpha \langle\psi|K_\alpha \rho K_\alpha^\dagger|\psi\rangle = \sum_\alpha \sum_n \lambda_n \langle\psi|K_\alpha|n\rangle \underbrace{\langle n|K_\alpha^\dagger|\psi\rangle}_{(\langle\psi|K_\alpha|n\rangle)^*}$$

$$= \sum_\alpha \sum_n \underbrace{\lambda_n}_{\geq 0} \underbrace{|\langle\psi|K_\alpha|n\rangle|^2}_{\geq 0} \geq 0$$

$\rightsquigarrow \rho'$ ist auch eine DM

$$\boxed{\rho(t+\Delta t) = \sum_\alpha K_\alpha(t,\Delta t)\, \rho(t)\, K_\alpha^\dagger(t,\Delta t)}$$

- oft benutzt in Quanten-Info
- Problem: Wie bekomme ich die K_α
- unhandlich

1.2. Lindblad Mastergleichung

auch LGKS für Lindblad - Gorini - Kossakowski - Sudarshan $\mathcal{O}(1976)$

allg. DGL 1. Ordnung ("Mastergleichung") für ρ_S mit konstanten Koeffizienten

$$\dot\rho_S = -i[H, \rho_S] + \sum_{\alpha\beta=1}^{N^2-1} \gamma_{\alpha\beta}\left[A_\alpha \rho_S A_\beta^\dagger - \frac{1}{2} A_\beta^\dagger A_\alpha \rho_S - \frac{1}{2} \rho_S A_\beta^\dagger A_\alpha\right]$$

$\rho = N\times N$-DM $\quad H = H^\dagger \qquad \gamma_{\alpha\beta} = \gamma_{\beta\alpha}^* \qquad \sum_{\alpha\beta} x_\alpha^* \gamma_{\alpha\beta} x_\beta \geq 0$

$$= -i[H, \rho_S] + \sum_{\alpha=1}^{N^2-1}\left[L_\alpha \rho_S L_\alpha^\dagger - \frac{1}{2}\{L_\alpha^\dagger L_\alpha, \rho_S\}\right]$$

Lindblad ME

- $\mathrm{Tr}\{\dot\rho_S\} = -i\,\mathrm{Tr}\{H\rho_S - \rho_S H\} + \sum_\alpha \mathrm{Tr}\{L_\alpha \rho_S L_\alpha^\dagger - \frac{1}{2} L_\alpha^\dagger L_\alpha \rho - \frac{1}{2}\rho L_\alpha^\dagger L_\alpha\}$

$\underline{N < \infty}$

- $(\dot\rho_S)^\dagger = \dot\rho_S \qquad$ wegen $(-iH\rho_S + i\rho_S H)^\dagger = +i\rho_S H - iH\rho_S$

und $\left(L_\alpha \rho_S L_\alpha^\dagger - \frac{1}{2} L_\alpha^\dagger L_\alpha \rho_S - \frac{1}{2}\rho_S L_\alpha^\dagger L_\alpha\right)^\dagger$

$= \left(L_\alpha \rho_S L_\alpha^\dagger - \frac{1}{2}\rho_S L_\alpha^\dagger L_\alpha - \frac{1}{2} L_\alpha^\dagger L_\alpha \rho_S\right)$

∘ $\varrho(t) = \sum_i \lambda_i(t) \, |\psi_i(t)\rangle\langle\psi_i(t)|$

EW: $\lambda_i \in \mathbb{R}$ $\qquad$ EV

$\langle\psi_i(t)|\psi_k(t)\rangle = \delta_{ki}$

$\langle\dot\psi_i|\psi_k\rangle + \langle\psi_i|\dot\psi_k\rangle = 0$

$$\dot\varrho = \sum_j \left[\dot\lambda_j \, |\psi_j\rangle\langle\psi_j| + \lambda_j \, |\dot\psi_j\rangle\langle\psi_j| + \lambda_j \, |\psi_j\rangle\langle\dot\psi_j| \right]$$

$$\langle\psi_i|\dot\varrho|\psi_i\rangle = \dot\lambda_i + \lambda_i \Big(\underbrace{\langle\psi_i|\dot\psi_i\rangle + \langle\dot\psi_i|\psi_i\rangle}_{=0} \Big)$$

$$\dot\lambda_i = \cancel{-i\langle\psi_i|H|\psi_i\rangle\lambda_i} + \cancel{i\lambda_i\langle\psi_i|H|\psi_i\rangle}$$

$$+ \sum_\alpha \left[\langle\psi_i| L_\alpha \Big(\sum_j \lambda_j |\psi_j\rangle\langle\psi_j|\Big) L_\alpha^\dagger |\psi_i\rangle - \langle\psi_i| L_\alpha^\dagger L_\alpha |\psi_i\rangle \cdot \lambda_i \right]$$

$$= \sum_j \Big(\sum_\alpha |\langle\psi_i|L_\alpha|\psi_j\rangle|^2 \Big) \lambda_j - \Big(\sum_j \sum_\alpha |\langle\psi_j|L_\alpha|\psi_i\rangle|^2 \Big) \lambda_i$$

$$= \sum_j R_{j\to i}(t) \, \lambda_j(t) - \sum_j R_{i\to j} \, \lambda_i(t) = \sum_j R_{ij}(t) \cdot \lambda_j$$

$$R_{ij}(t) = \begin{cases} R_{j\to i}(t) & : j \neq i \\ -\sum_{k\neq i} R_{i\to k}(t) & : i = j \end{cases} \qquad \leadsto \sum_i \dot\lambda_i = 0$$

$$R_{j\to i}(t) = \sum_\alpha |\langle\psi_i|L_\alpha|\psi_j\rangle|^2 \geq 0$$

$t = 0$: $\lambda_i(0) \geq 0$

sei $\lambda_{\bar i}(t^*) = 0$ $\leadsto$ $\frac{d}{dt}\lambda_{\bar i}\Big|_{t=t^*} = \sum_n R_{\bar i n} \, \lambda_n\Big|_{t=t^*} = \sum_{n\neq\bar i} \underbrace{R_{\bar i n}}_{\geq 0} \underbrace{\lambda_n}_{\geq 0}$

und $\lambda_{n\neq\bar i}(t^*) \geq 0$ $\qquad \geq 0$

1.2.2 Beispiel: Kavität im thermischen Bad

spontane Emission, induzierte Emission, Absorption

$$\dot\varrho = -i[\Omega a^\dagger a, \varrho] + \Gamma(1+n_B)\left[a\varrho a^\dagger - \tfrac{1}{2}\{a^\dagger a, \varrho\}\right]$$
$$+ \Gamma \cdot n_B \left[a^\dagger \varrho a - \tfrac{1}{2}\{a a^\dagger, \varrho\}\right]$$

$\Omega > 0$

$L_1 = \sqrt{\Gamma(1+n_B)} \; a$ $\qquad$ $\Gamma > 0$ Kopplung an das Reservoir

$L_2 = \sqrt{\Gamma \cdot n_B} \cdot a^\dagger$ $\qquad$ $n_B = \frac{1}{e^{\beta\Omega}-1} > 0$ $(\mu = 0)$

$a|n\rangle = \sqrt{n}\,|n-1\rangle$

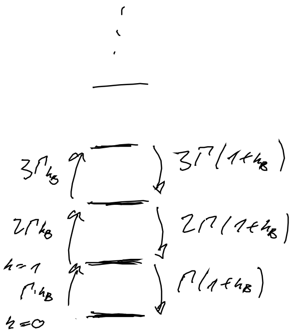
$a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$

$$\langle n|\dot\rho|n\rangle \equiv \dot\rho_n = \cancel{-i\Omega\langle n|a^\dagger a\,\rho|n\rangle} + \cancel{i\Omega\langle n|\rho\, a^\dagger a|n\rangle}$$

$$+\Gamma(1+n_B)\left((n+1)\,\rho_{n+1} - \tfrac{1}{2}\cdot 2n\cdot\rho_n\right)$$

$$+\Gamma\cdot n_B\left(n\,\rho_{n-1} - (1+n)\cdot\rho_n\right)$$

$aa^\dagger = 1 + a^\dagger a$



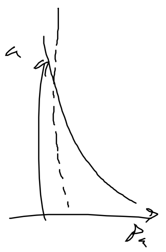
$$\dot\rho_n = \Gamma\cdot n_B\cdot\rho_{n-1}\cdot n + \Gamma(1+n_B)(n+1)\,\rho_{n+1}$$

$$-\Gamma\left(n+(2n+1)\cdot n_B\right)\rho_n$$

$\underline{n=0}$ $\quad \dot\rho_0 = \Gamma(1+n_B)\,\rho_1 - \Gamma n_B\cdot\rho_0 \stackrel{!}{=} 0 \quad$ (SS)

$$\lim_{t\to\infty}\rho = \bar\rho \leadsto \frac{\bar\rho_1}{\bar\rho_0} = \frac{n_B}{1+n_B} = e^{-\beta\Omega}$$

$$\leadsto \frac{\bar\rho_n}{\bar\rho_{n-1}} = e^{-\beta\Omega}$$



$$\bar\rho_n = \alpha\cdot e^{-\beta\cdot n\cdot\Omega} \qquad \sum_n \bar\rho_n = 1 \leadsto \alpha = (1-e^{-\beta\Omega})$$

$$\bar\rho_{nm} = \delta_{nm}\cdot(1-e^{-\beta\Omega})\cdot e^{-\beta\cdot n\cdot\Omega}$$

$$\bar\rho = \frac{e^{-\beta H}}{\mathrm{Tr}\{e^{-\beta H}\}} \qquad H = \Omega a^\dagger a$$