WdH

- Lösung der MG durch gekopp. Observablen

$$\mathrm{Tr}\{\hat O_i \dot\rho\} = -i\,\mathrm{Tr}\{O_i(H\rho - \rho H)\} + \sum_\alpha \mathrm{Tr}\{O_i[L_\alpha \rho L_\alpha^\dagger - \tfrac12 L_\alpha^\dagger L_\alpha \rho - \tfrac12 \rho L_\alpha^\dagger L_\alpha]\}$$

$$\frac{d}{dt}\langle \hat O_i\rangle = \underbrace{+i\langle[H,O_i]\rangle + \sum_\alpha \langle (L_\alpha^\dagger O_i L_\alpha - \tfrac12\{O_i, L_\alpha^\dagger L_\alpha\})\rangle}_{\sum_j g_{ij}\langle \hat O_j\rangle}$$

"duale Mastergleichung" analog zum Heisenberg-Bild

wirkt nur auf W

- partielle Spur

$$A = \sum_\alpha V_\alpha \otimes W_\alpha \longrightarrow \mathrm{Tr}_W\{A\} = \sum_\alpha V_\alpha\, \mathrm{Tr}\{W_\alpha\}$$

wirkt nur auf V

$$\rho_{AB} \longrightarrow \rho_A = \mathrm{Tr}_B\{\rho_{AB}\}, \quad \rho_B = \mathrm{Tr}_A\{\rho_{AB}\}$$

falls $\rho_{AB} = \bar\rho_A\otimes\bar\rho_B$

$\longrightarrow \mathrm{Tr}_A\{\rho_{AB}\} = \bar\rho_B$ & $\mathrm{Tr}_B\{\rho_{AB}\}=\bar\rho_A$

im Allgemeinen $\rho_{AB} \neq \rho_A\otimes\rho_B$ (Verschränkung)

$\rho_A = \mathrm{Tr}_B\{\rho_{AB}\}$

$\rho_B = \mathrm{Tr}_A\{\rho_{AB}\}$

Bsp.: $|\Psi_{AB}\rangle = \frac{1}{\sqrt2}[|0\rangle_A\otimes|1\rangle_B + |1\rangle_A\otimes|0\rangle_B]$

$\rho_{AB} = |\Psi_{AB}\rangle\langle\Psi_{AB}|$

$\rho_A = \frac12[|0\rangle\langle0| + |1\rangle\langle 1|] \neq \rho_A^2$

$\rho_A\otimes\rho_B \neq \rho_{AB}$

$A \hat= $ System $\dim(A)$ klein

$B \hat= $ Reservoir $\dim(B) \longrightarrow \infty$

- WW-Bild

$$\dot\rho_{tot} = -i[\underbrace{H_S+H_B}_{H_0}+H_I, \rho_{tot}]$$

$$\boxed{\dot{\tilde\rho}_{tot} = -i[\tilde H_I(t), \tilde\rho_{tot}(t)]}$$

$$\tilde\rho = e^{+i(H_S+H_B)t}\rho(t)\, e^{-i(H_S+H_B)t}$$

$$H_I = \sum_\alpha A_\alpha\otimes B_\alpha$$

$$\tilde H_I(t) = e^{+i(H_S+H_B)t} H_I\, e^{-i(H_S+H_B)t} \quad \left(\text{o.B.d.A. } A_\alpha = A_\alpha^\dagger,\ B_\alpha = B_\alpha^\dagger \text{ möglich}\right)$$

$$= \sum_\alpha e^{+iH_St}A_\alpha e^{-iH_St}\otimes e^{+iH_Bt}B_\alpha e^{-iH_Bt}$$

- formal integrieren $\int_0^t[\ldots]dt'$

$$\tilde\rho_{tot}(t) = \rho_{tot}(0) - i\int_0^t[\tilde H_I(t'),\tilde\rho_{tot}(t')]\,dt'$$

wieder einsetzen

$$\dot{\tilde\rho}_{tot} = -i[\tilde H_I(t), \rho_{tot}(0)] - \int_0^t dt'\,[\tilde H_I(t),[\tilde H_I(t'),\tilde\rho_{tot}(t')]]$$ immer noch exakt

$\mathrm{Tr}_B\{\ldots\}$

$$\dot{\tilde\rho}_S = -i\,\mathrm{Tr}_B\{[\tilde H_I(t),\rho_{tot}(0)]\} - \int_0^t dt'\, \mathrm{Tr}_B\{[\tilde H_I(t),[\tilde H_I(t'),\tilde\rho_{tot}(t')]]\}$$

(1) Born'sche Näherung $\tilde\rho_{tot}(t) = \rho_S(t)\otimes\bar\rho_B + \mathcal O(\lambda)$ $: \tilde H_I = \mathcal O(\lambda)$

$\rho_S^0 = \rho_S(0)$

$$\dot{\tilde\rho}_S = \underbrace{-i\,\mathrm{Tr}_B\{[\tilde H_I(t),\rho_S^0\otimes\bar\rho_B]\}}_{\mathcal O(\lambda)} - \underbrace{\int_0^t dt'\,\mathrm{Tr}_B\{[\tilde H_I(t),[\tilde H_I(t'),\rho_S(t')\otimes\bar\rho_B]]\}}_{\substack{\text{Nicht-Markov'sche Mastergleichung} \\ \mathcal O(\lambda^2)}} + \mathcal O(\lambda^3)$$

kann zum Verschwinden gebracht werden

$$\mathrm{Tr}_B\left\{\sum_\alpha \left[\tilde{A}_\alpha \otimes \tilde{B}_\alpha, \rho_S^0 \otimes \bar{\rho}_B\right]\right\} = \sum_\alpha \left[\tilde{A}_\alpha \cdot \rho_S^0 \, \mathrm{Tr}\{\tilde{B}_\alpha \bar{\rho}_B\} - \rho_S^0 \tilde{A}_\alpha \, \mathrm{Tr}\{\bar{\rho}_B \tilde{B}_\alpha\}\right]$$

$$= \sum_\alpha \left[\tilde{A}_\alpha, \rho_S^0\right] \cdot \underbrace{\mathrm{Tr}_B\left\{e^{+iH_B t} B_\alpha e^{-iH_B t} \bar{\rho}_B\right\}}_{= \mathrm{Tr}_B\{B_\alpha \bar{\rho}_B\}}$$

falls $[H_B, \bar{\rho}_B] = 0$
Res. ist im GG

$$H_S \to H_S' = H_S + \sum_\alpha g_\alpha A_\alpha \qquad g_\alpha \in \mathbb{R}$$

$$H_I \to H_I' = \sum_\alpha A_\alpha \otimes \underbrace{(B_\alpha - g_\alpha \cdot \mathbb{1})}_{B_\alpha'} \qquad g_\alpha = \mathrm{Tr}_B\{B_\alpha \bar{\rho}_B\}$$

umschreiben

$$\boxed{C_{\alpha\beta}(t_1, t_2) = \mathrm{Tr}_B\left\{\tilde{B}_\alpha(t_1) \tilde{B}_\beta(t_2) \bar{\rho}_B\right\}}$$

Bad-Korrelations-Funktion

$[H_B, \bar{\rho}_B] = 0$

$$\Downarrow$$

$$= \mathrm{Tr}_B\left\{e^{-iH_B t_2} \tilde{B}_\alpha(t_1) e^{+iH_B t_2} B_\beta \bar{\rho}_B\right\}$$

$$= \mathrm{Tr}_B\left\{\tilde{B}_\alpha(t_1 - t_2) B_\beta \bar{\rho}_B\right\} = C_{\alpha\beta}(t_1 - t_2)$$

$$\dot{\rho}_S = -\sum_{\alpha\beta} \int_0^t dt' \left\{C_{\alpha\beta}(t, t') \left[\tilde{A}_\alpha(t), \tilde{A}_\beta(t') \rho_S(t')\right] + C_{\beta\alpha}(t', t) \left[\rho_S(t') A_\beta(t'), A_\alpha(t)\right]\right\}$$

Nicht-Markovsche Mastergleichung $\qquad \dot{\rho}_S = \int_0^t \mathcal{W}(t, t') \rho_S(t') dt'$

- geschlossene Integro-DGL
- nur für einfache $C_{\alpha\beta}(t, t')$ lösbar
- erhält $\mathrm{Tr}\{\rho_S\}$ & $\rho_S = \rho_S^\dagger$

(2) Markov Näherung(en)

$$C_{\alpha\beta}(t_1 - t_2) = C_{\alpha\beta}(\tau) \quad \text{fallen schnell ab}$$

$$\dot{\rho}_S \approx \int_0^t W(t, t') dt' \, \rho_S(t) \approx \int_0^\infty W(t, t') dt' \, \rho_S(t)$$

1. Markov-Näherung $\qquad$ 2. Markov-Näherung

spont. Emission $\qquad h_k \in \mathbb{C}$

Bsp. $H_I = A \otimes \sum_k \left(h_k b_k + h_k^* b_k^\dagger\right)$

System $\qquad$ Bosonen

$$C_{11}(\tau) = \mathrm{Tr}_B\left\{\underbrace{\sum_k \left(h_k b_k e^{-i\omega_k \tau} + h_k^* b_k^\dagger e^{+i\omega_k \tau}\right)}_{\tilde{B}_1(\tau)} \left(\sum_q h_q b_q + h_q^* b_q^\dagger\right) \frac{e^{-\beta \sum_k \omega_k b_k^\dagger b_k}}{Z_B}\right\}$$

$$= \sum_k |h_k|^2 \left[\underline{\langle b_k b_k^\dagger \rangle} e^{-i\omega_k \tau} + \underline{\langle b_k^\dagger b_k \rangle} e^{+i\omega_k \tau} \right]$$

$$\frac{e^{-\beta \sum_k \omega_k b_k^\dagger b_k}}{Z} = \prod_k \frac{e^{-\beta \omega_k b_k^\dagger b_k}}{Z_k}$$

- $\mathrm{Tr}\left\{ b_k e^{-\beta\omega_k b_k^\dagger b_k} \right\} = \sum_{n=0}^\infty \langle n | b_k e^{-\beta \omega_k b_k^\dagger b_k} | n \rangle = \sum_{n=0}^\infty \underbrace{\langle n | b_k | n \rangle}_{0} e^{-\beta\omega_k \cdot n} = 0$
- $\mathrm{Tr}\left\{ b_k^2 e^{-\beta\omega_k b_k^\dagger b_k} \right\} = 0$
- $\mathrm{Tr}\left\{ b_k^\dagger b_k \frac{e^{-\beta\omega_k b_k^\dagger b_k}}{Z_k} \right\} = \sum_{n=0}^\infty n \cdot \frac{e^{-\beta\omega_k \cdot n}}{Z_k} = -\sum_{n=0}^\infty \frac{\partial}{\partial(\beta\omega_k)} \ln \underbrace{\sum_{n=0}^\infty e^{-\beta\omega_k \cdot n}}_{Z_k}$

$$= \frac{\partial}{\partial(\beta\omega_k)} \cdot \ln\left(1 - e^{-\beta\omega_k}\right) \qquad \frac{1}{1-e^{-\beta\omega_k}}$$

$$= \frac{e^{-\beta\omega_k}}{1-e^{-\beta\omega_k}} = \frac{1}{e^{\beta\omega_k} - 1} = n_B(\omega_k)$$

- $\mathrm{Tr}\{ b_k b_k^\dagger \rho_B \}$
$= \mathrm{Tr}\{ (\mathbb{1} + b_k^\dagger b_k) \rho_B \}$
$= 1 + n_B(\omega_k)$

$$C_{11}(\tau) = \sum_k |h_k|^2 \left[(1 + n_B(\omega_k)) e^{-i\omega_k\tau} + n_B(\omega_k) e^{+i\omega_k\tau} \right]$$

$$= \frac{1}{2\pi} \int_0^\infty \Gamma(\omega) \left[(1 + n_B(\omega)) e^{-i\omega\tau} + n_B(\omega) e^{+i\omega\tau} \right] d\omega$$

spektrale (Kopplungs-) Dichte

$$\Gamma(\omega) = 2\pi \sum_k \underbrace{|h_k|^2}_{\text{Kopplungen}} \delta(\omega - \underbrace{\omega_k}_{\geq 0}) \geq 0$$

Freq. des H_B

$n_B(\omega) = -[1 + n_B(-\omega)]$ prüfen

analytisch fortsetzen

$\tilde{\Gamma}(-\omega) = -\tilde{\Gamma}(+\omega)$

$$\Rightarrow \frac{1}{2\pi} \int_{-\infty}^{+\infty} \underbrace{\tilde{\Gamma}(\omega)[1 + n_B(\omega)]}_{\gamma_{11}(\omega) = \int C(\tau) e^{+i\omega\tau} d\tau} e^{-i\omega\tau}$$

$$\gamma_{11}(\omega) = \tilde{\Gamma}(\omega)[1 + n_B(\omega)] \geq 0$$

falls $\gamma_{11}(\omega)$ flach

$\Rightarrow$ Markov-Näherung ist gut

$$\dot{\tilde{\rho}}_S = -\int_0^\infty \mathrm{Tr}_B\left\{ \left[\tilde{H}_I(t), \left[\tilde{H}_I(t-\tau), \tilde{\rho}_S(t) \otimes \bar{\rho}_B \right] \right] \right\} d\tau$$

im Schröd-Bild

Badkopplung — Systemkopplung — $(A_\alpha = A_\alpha^\dagger)$

$$\boxed{\dot{\rho}_S = -i[H_S, \rho] - \left[\sum_{\alpha\beta} \int_0^\infty C_{\alpha\beta}(\tau) \left[A_\alpha, e^{-iH_S\tau} A_\beta e^{+iH_S\tau} \rho_S(t) \right] d\tau + \text{h.c.} \right]}$$

- zeit-lokal $\qquad \dot{\rho}_S = \mathcal{L}\rho_S \qquad$ "Redfield-Gleichung" ((oft Markovsche Mastergl.))
- $\mathrm{Tr}\{\rho_S\} = 1 \qquad \rho_S = \rho_S^\dagger$
- leider i.A. keine Lindblad-Form

③ Säkular-Näherung

notwendig um immer eine Lindblad-Form zu erhalten $\rightarrow$ nächste Woche