English Summary:

Maxwell's equations in vacuum

$$\boxed{\begin{aligned} &\nabla\times \underline{E} + \dot{\underline{B}} = 0 \\ &\nabla\cdot \underline{B} = 0 \\ &\nabla\cdot \underline{D} = \rho \\ &\nabla\times \underline{H} - \dot{\underline{D}} = \underline{j} \end{aligned}}$$ diff. form

$$\boxed{\begin{aligned} &\oint_{\partial F} \underline{E}\cdot d\underline{s} = -\frac{\partial}{\partial t}\underbrace{\int_F \underline{B}\cdot d\underline{f}}_{\Phi} \\ &\oint_{\partial V} \underline{B}\cdot d\underline{f} = 0 \\ &\oint_{\partial V} \underline{D}\cdot d\underline{f} = Q \\ &\oint_{\partial F} \underline{H}\cdot d\underline{s} = \int_F \dot{\underline{D}}\cdot d\underline{f} + I \end{aligned}}$$ integral form

Φ magn. flux

$\underline{D}(\underline{r},t) = \epsilon_0 \underline{E}(\underline{r},t)$ dielectric displacement

$\underline{H}(\underline{r},t) = \frac{1}{\mu_0}\underline{B}(\underline{r},t)$ magnetic field

3.3 TCP – Invarianz

Invarianzeigenschaften der Maxwell-Gln.:

- Lorentz-Invarianz (s. Kap. 6)
- TCP-Invarianz

außerdem:

- linear in $\underline{E}, \underline{B}, \underline{D}, \underline{H}$ (Superpositionsprinzip!)
- 1. Ordnung in t (Kausalitätsprinzip: $\underline{E}, \underline{B}, \underline{D}, \underline{H}$ zur Zeit $t=0$ soll den Zustand für $t>0$ vollständig festlegen!)

Zeitumkehr $T: t \to t' = -t$

Ladungsumkehr $C: Q \to Q' = -Q$

Paritätsumkehr $P: \underline{r} \to \underline{r}' = -\underline{r}$

(i) Zeitumkehr – Transformation

$$T_g := \{T\text{-invariante Observable } A : TA = A\}$$
$$= \{\underline{r}, d\underline{r}, \underline{a} := \frac{d^2\underline{r}}{dt^2}, m, q, \rho = \lim_{\Delta V\to 0}\frac{\Delta q}{\Delta V},$$
$$\underline{F} = m\underline{a}, \underline{E} = \frac{1}{q}\underline{F}, \text{ Potential } \phi, \dots\}$$

„gerade" unter T

$$T_u := \{A : TA = -A\}$$ „ungerade" unter T

$$= \{\underline{v} = \frac{d\underline{r}}{dt}, \underline{j} = \rho\underline{v}, \underline{B}, \underline{A}, \dots\}$$

denn: $\underset{T_g}{\underline{F}} = \underset{T_g}{q}\,\underset{T_u}{\underline{v}}\times\underline{B}$, $\underset{T_u}{\underline{B}} = \nabla\times\underset{T_g}{\underline{A}}$

$\Rightarrow$ T-Invarianz der Maxwell-Gln.:

$$T: \{\nabla\times\underline{E} + \dot{\underline{B}} = 0\} \longrightarrow \{\nabla\times\underline{E} + \dot{\underline{B}} = 0\}$$

$$T: \{\nabla\cdot\underline{B} = 0\} \longrightarrow \{-\nabla\cdot\underline{B} = 0\}$$

$$T: \{\epsilon_0\nabla\cdot\underline{E} = \varrho\} \longrightarrow \{\epsilon_0\nabla\cdot\underline{E} = \varrho\}$$

$$T: \{\nabla\times\underline{H} - \dot{\underline{D}} = \underline{j}\} \longrightarrow \{-\nabla\times\underline{H} + \dot{\underline{D}} = -\underline{j}\}$$

Kontinuitätsgl.:

$$T: \{\tfrac{\partial\varrho}{\partial t} + \nabla\cdot\underline{j} = 0\} \longrightarrow \{-\tfrac{\partial\varrho}{\partial t} - \nabla\cdot\underline{j} = 0\}$$

Die Maxwell-Gln. sind <u>forminvariant</u>!

(ii) <u>Ladungsumkehr</u> (-konjugation)

$C_g = \{\underline{F}, m, \underline{r}, \underline{v}, \underline{a}, \dots\}$ <u>gerade</u> unter C

$C_u = \{\underline{E} = \frac{1}{q}\underline{F}, \underline{B}, \varrho, \underline{j}, \dots\}$ <u>ungerade</u> unter C

da $\underline{F} = q\underline{v}\times\underline{B}$

$\Rightarrow$ C-Invarianz der Maxwell-Gln.:

$$C: \{\nabla\times\underline{E} + \dot{\underline{B}} = 0\} \longrightarrow \{-\nabla\times\underline{E} - \dot{\underline{B}} = 0\}$$

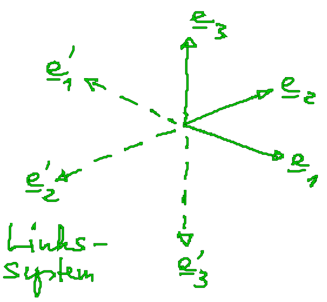
$$C: \{\nabla\cdot\underline{B} = 0\} \longrightarrow \{-\nabla\cdot\underline{B} = 0\}$$

$$C: \{\epsilon_0\nabla\cdot\underline{E} = \varrho\} \longrightarrow \{-\epsilon_0\nabla\cdot\underline{E} = -\varrho\}$$

$$C: \{\nabla\times\underline{H} - \dot{\underline{D}} = \underline{j}\} \longrightarrow \{-\nabla\times\underline{H} + \dot{\underline{D}} = -\underline{j}\}$$

$$C: \{\tfrac{\partial\varrho}{\partial t} + \nabla\cdot\underline{j} = 0\} \longrightarrow \{-\tfrac{\partial\varrho}{\partial t} - \nabla\cdot\underline{j} = 0\}$$

(iii) <u>Paritätsumkehr</u> (Räumliche Spiegelung am Ursprung, Inversion, Vertauschung rechts $\Leftrightarrow$ links)



$P\underline{r} = -\underline{r}$ „<u>polarer</u>" Vektor

aber $P(\underline{a}\times\underline{b}) = (-\underline{a})\times(-\underline{b}) = \underline{a}\times\underline{b}$ „<u>axialer</u>" Vektor

<u>P-invariant</u> (Pseudovektor)

Seien $\underline{a}, \underline{b}$ polar $\Rightarrow$ $\underline{a} \times \underline{\omega}$ polar

$\underline{\omega}, \sigma$ axiale $\quad$ $\underline{a} \times \underline{b}$, $\underline{\omega} \times \underline{\sigma}$ axial

$\underline{a} \cdot \underline{b}$ skalar $\quad P(\underline{a} \cdot \underline{b}) = \underline{a} \cdot \underline{b}$

$\underline{a} \cdot \underline{\omega}$ pseudoskalar $\quad P(\underline{a} \cdot \underline{\omega}) = -\underline{a} \cdot \underline{\omega}$

$P_g = \{$ Skalare m, q; axiale Vektoren $\underline{B}$ $\}$ gerade

$$\underset{P_u}{\underline{F}} = \underset{P_g}{q}\, \underset{P_u}{\underline{v}} \times \underset{P_g}{\underline{B}}$$

$P_u = \{$ polare Vektoren $\underline{r}, d\underline{r}, \underline{v}, \underline{a}, \underline{F}, \underline{E} = \frac{1}{q}\underline{F}, \underline{j} = \rho \underline{v}, \underline{A}$;

Pseudoskalar $\underset{u}{\underline{\nabla}} \cdot \underset{g}{\underline{B}}$ $\}$

$$\underset{g}{\underline{B}} = \underset{u}{\underline{\nabla}} \times \underset{u}{\underline{A}}$$

P-Invarianz der Maxwell-Gln.:

$P: \{ \underline{\nabla} \times \underline{E} + \dot{\underline{B}} = 0 \} \rightarrow \{ \underline{\nabla} \times \underline{E} + \dot{\underline{B}} = 0 \}$

$P: \{ \underline{\nabla} \cdot \underline{B} = 0 \} \rightarrow \{ -\underline{\nabla} \cdot \underline{B} = 0 \}$!

$P: \{ \epsilon_0 \underline{\nabla} \cdot \underline{E} = \rho \} \rightarrow \{ \epsilon_0 \underline{\nabla} \cdot \underline{E} = \rho \}$

$P: \{ \underline{\nabla} \times \underline{H} - \dot{\underline{D}} = \underline{j} \} \rightarrow \{ -\underline{\nabla} \times \underline{H} + \dot{\underline{D}} = \underline{j} \}$

$P: \{ \frac{\partial}{\partial t}\rho + \underline{\nabla} \cdot \underline{j} = 0 \} \rightarrow \{ \frac{\partial}{\partial t}\rho + \underline{\nabla} \cdot \underline{j} = 0 \}$

NB: Schwache WW verletzt die Paritätserhaltung (Pauli 1957)
(P, PC, C, T verletzt, aber PCT ist erhalten)

Lorentzkraft $\quad \underline{F} = q(\underline{E} + \underline{v} \times \underline{B})$

soll aus einem Extremal-Prinzip (Hamilton'sches Prinzip) ableitbar sein.

$\Rightarrow$ Zus.hang zwischen $\underline{E}, \underline{B}$ und $\phi, \underline{A}$

Suche Lagrange-Fkt. $L(\underline{r}, \underline{v}, t)$, so dass Lagrange-Gln.

$$\frac{d}{dt}\frac{\partial L}{\partial v_k} - \frac{\partial L}{\partial x_k} = 0$$

ergibt: $\quad m\ddot{\underline{r}} = q\left[\underline{E}(\underline{r},t) + \underline{v} \times \underline{B}(\underline{r},t)\right] \quad (*)$

Lagrange-Fkt. $\boxed{L = \frac{m}{2}v^2 + q\left[\underline{v} \cdot \underline{A} - \phi(\underline{r},t)\right]}$

Tatsächlich gilt

$$p_k = \frac{\partial L}{\partial v_k} = \underset{\text{kinet. Impuls}}{m v_k} + q A_k(\underline{r},t) = \text{„}\underline{\text{kanon. Impuls}}\text{"}$$

$$\frac{d}{dt}\frac{\partial L}{\partial v_k} = m\ddot{x}_k + q \frac{d}{dt} A_k(\underline{r},t)$$

totale Zeitableit. längs einer Bahn $\underline{r}(t)$

$$= m\ddot{x}_k + q\left(\frac{\partial}{\partial t} A_k + \sum_\ell \frac{\partial A_k}{\partial x_\ell}\dot{x}_\ell\right)$$

$$= m\ddot{x}_k + q\left(\frac{\partial}{\partial t} + \underline{v}.\underline{\nabla}\right) A_k$$

$$\frac{\partial L}{\partial x_k} = q\left[\frac{\partial}{\partial x_k}(\underline{v}.\underline{A}) - \frac{\partial}{\partial x_k}\phi\right]$$

$$\Rightarrow \quad 0 = \frac{d}{dt}\frac{\partial L}{\partial v_k} - \frac{\partial L}{\partial x_k} = m\ddot{x}_k + q\frac{\partial}{\partial t}A_k + q\underbrace{\left[(\underline{v}.\underline{\nabla})A_k - \frac{\partial}{\partial x_k}(\underline{v}.\underline{A})\right]}_{-[\underline{v}\times(\underline{\nabla}\times\underline{A})]_k} + q\frac{\partial}{\partial x_k}\phi$$

$$0 = m\ddot{\underline{r}} + q\left[\frac{\partial}{\partial t}\underline{A} + \underline{\nabla}\phi - \underline{v}\times(\underline{\nabla}\times\underline{A})\right]$$

Vergleich mit Lorentzkraft (*) :

$$\boxed{\begin{aligned} \underline{E}(\underline{r},t) &= -\frac{\partial}{\partial t}\underline{A}(\underline{r},t) - \underline{\nabla}\phi(\underline{r},t) \\ \underline{B}(\underline{r},t) &= \underline{\nabla}\times\underline{A}(\underline{r},t) \end{aligned}}$$

damit sind die homog. Maxwell-Gln. automatisch erfüllt!

$$\underline{\nabla}.\underline{B} = \underline{\nabla}.(\underline{\nabla}\times\underline{A}) \equiv 0$$

$$\underline{\nabla}\times\underline{E} = -\frac{\partial}{\partial t}\underbrace{\underline{\nabla}\times\underline{A}}_{\underline{B}} - \underbrace{\underline{\nabla}\times(\underline{\nabla}\phi)}_{\equiv 0}$$

3.4 Energiebilanz

Die Maxwell-Gln. enthalten die Kontin.gl. für ϱ :

$$\frac{\partial}{\partial t}\varrho + \underline{\nabla}.\underline{j} = 0 \qquad (\text{Ladungserhaltung})$$

<u>Frage</u>: weitere <u>Erhaltungssätze</u> für <u>extensive</u> phys. Obs., z.B. Energie, Impuls, Drehimpuls ?
(Ext.: additiv bei Systemzus.setzung)
Intensiv: z.B. Temp.

<u>Energietransport durch das el. magn. Feld</u> :

$$\underline{\nabla} \times \underline{E} + \frac{\partial}{\partial t}\underline{B} = 0 \qquad | \cdot \underline{H}$$

$$\underline{\nabla} \times \underline{H} - \frac{\partial}{\partial t}\underline{D} = \underline{j} \qquad | \cdot \underline{E}$$

$$\Rightarrow \underbrace{\underline{H}\cdot(\underline{\nabla}\times\underline{E}) - \underline{E}\cdot(\underline{\nabla}\times\underline{H})}_{\underline{\nabla}\cdot(\underline{E}\times\underline{H})} + \underbrace{\underline{H}\frac{\partial}{\partial t}\underline{B}}_{\frac{1}{\mu_0}\underbrace{\underline{B}\frac{\partial}{\partial t}\underline{B}}_{\frac{1}{2}\frac{\partial}{\partial t}\underline{B}^2}} + \underbrace{\underline{E}\frac{\partial}{\partial t}\underline{D}}_{\epsilon_0\underbrace{\underline{E}\frac{\partial}{\partial t}\underline{E}}_{\frac{1}{2}\frac{\partial}{\partial t}\underline{E}^2}} = -\underline{j}\cdot\underline{E}$$

$$\Rightarrow \boxed{\frac{\partial}{\partial t}w + \underline{\nabla}\cdot\underline{S} = -\underline{j}\cdot\underline{E}}$$ Kontin. gl.

= Bilanzgl. für Energietransport

$$w := \frac{\epsilon_0}{2}\underline{E}^2 + \frac{1}{2\mu_0}\underline{B}^2 = \frac{1}{2}(\underline{E}\cdot\underline{D} + \underline{B}\cdot\underline{H})$$ Energiedichte

$$\underline{S} := \underline{E}\times\underline{H}$$ Energiestromdichte des el. magn. Feldes

(Poynting – Vektor)

$$\sigma = -\underline{j}\cdot\underline{E}$$ Quelldichte der Feldenergie

(Leistungsdichte)

$$\left.\begin{array}{ll} \underline{j}\cdot\underline{E} > 0 & \text{Abnahme} \\ \underline{j}\cdot\underline{E} < 0 & \text{Zunahme} \end{array}\right\}$$ der Feldenergie