English Summary:

Transformation of current and fields

4-current density $\boxed{j^\mu := (c\rho, \underline{j})}$

charge conservation $\boxed{\partial_\mu j^\mu = 0}$ $\quad \partial_\mu = \frac{\partial}{\partial x^\mu}$

Lorentz invariant

d'Alembert op. $\boxed{\Box = -\partial_\mu \partial^\mu} = \Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$

wave eq. $\boxed{\partial_\mu \partial^\mu \phi^\nu = \frac{1}{\epsilon_0 c} j^\nu}$

4-potential $\phi^0 := \phi$, $\phi^i := cA^i$ $(i = 1,2,3)$

gauge transformation $\boxed{\tilde{\phi}^\mu = \phi^\mu + \partial^\mu \varphi(x^\nu)}$

scalar gauge fct.

Felder $\underline{E}, \underline{B}$

$$\underline{E} = -\underline{\nabla}\phi - \frac{\partial}{\partial t}\underline{A}$$

$$\Rightarrow E^i = -\partial_i \phi - \frac{1}{c}\frac{\partial}{\partial t} cA^i = -\partial_i \phi^0 - \partial_0 \phi^i = \partial^i \phi^0 - \partial^0 \phi^i$$

$(i = 1,2,3)$

$$\underline{B} = \underline{\nabla} \times \underline{A}$$

$$\Rightarrow cB^1 = \partial_2 cA^3 - \partial_3 cA^2 = \partial_2 \phi^3 - \partial_3 \phi^2 = \partial^3 \phi^2 - \partial^2 \phi^3$$

zyklische Vertauschung: $\quad cB^2 = \partial^1 \phi^3 - \partial^3 \phi^1$

$\quad cB^3 = \partial^2 \phi^1 - \partial^1 \phi^2$

Zusammenfassung antisymm. Feldtensor

$\boxed{F^{\mu\nu} := \partial^\mu \phi^\nu - \partial^\nu \phi^\mu}$, $\quad F^{\mu\nu} = -F^{\nu\mu}$ $\quad (\mu, \nu = 0,1,2,3)$

Wegen der Antisymm. hat $F^{\mu\nu}$ nur 6 unabhängige Komponenten:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -cB^3 & cB^2 \\ E^2 & cB^3 & 0 & -cB^1 \\ E^3 & -cB^2 & cB^1 & 0 \end{pmatrix}$$

Lorentz-Trafo der Felder

Als Tensor 2. Stufe transformiert sich $F^{\mu\nu}$ wie

$$F'^{\mu\nu} = \Lambda^\mu{}_\lambda \Lambda^\nu{}_\kappa F^{\lambda\kappa}, \quad \Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\beta\gamma & & \\ -\beta\gamma & \gamma & & 0 \\ & & 1 & \\ & 0 & & 1 \end{pmatrix}$$

Im einzelnen:

$$E'^1 = F'^{10} = \Lambda^1{}_\lambda \Lambda^0{}_\kappa F^{\lambda\kappa} = -\beta\gamma \Lambda^0{}_\kappa F^{0\kappa} + \gamma \Lambda^0{}_\kappa F^{1\kappa}$$

$$= (\beta\gamma)^2 F^{01} + (\gamma)^2 F^{10}$$

$$= \underbrace{\gamma^2(1-\beta^2)}_{1} F^{10} = E^1$$

$$E'^2 = F'^{20} = \Lambda^2{}_\lambda \Lambda^0{}_\kappa F^{\lambda\kappa} = \Lambda^0{}_\kappa F^{2\kappa} = \gamma F^{20} - \beta\gamma F^{21}$$

$$= \gamma(E^2 - vB^3)$$

$$E'^3 = F'^{30} = \Lambda^0{}_\kappa F^{3\kappa} = \gamma F^{30} - \beta\gamma F^{31}$$

$$= \gamma(E^3 + vB^2)$$

$$B'^1 = \frac{1}{c} F'^{32} = \frac{1}{c} \Lambda^3{}_\lambda \Lambda^2{}_\kappa F^{\lambda\kappa} = \frac{1}{c} F^{32} = B^1$$

$$B'^2 = \frac{1}{c} F'^{13} = \frac{1}{c} \Lambda^1{}_\lambda \Lambda^3{}_\kappa F^{\lambda\kappa} = \frac{1}{c} \Lambda^1{}_\lambda F^{\lambda 3} = -\frac{\beta\gamma}{c} F^{03} + \frac{\gamma}{c} F^{13}$$

$$= \gamma\left(B^2 + \frac{v}{c^2} E^3\right)$$

$$B'^3 = \gamma\left(B^3 - \frac{v}{c^2} E^2\right)$$

Zusammenfassung:

$$\boxed{\begin{aligned} E'^1 &= E^1 & B'^1 &= B^1 \\ E'^2 &= \gamma(E^2 - vB^3) & B'^2 &= \gamma\left(B^2 + \frac{v}{c^2}E^3\right) \\ E'^3 &= \gamma(E^3 + vB^2) & B'^3 &= \gamma\left(B^3 - \frac{v}{c^2}E^2\right) \end{aligned}}$$

Elektr. u. magnet. Felder werden beim Übergang zwischen verschiedenen Inertialsystemen in einander transformiert.

Umeichung

$$\tilde{\phi}^\mu = \phi^\mu + \partial^\mu \varphi$$

$$\Rightarrow \tilde{F}^{\mu\nu} = \partial^\mu \tilde{\phi}^\nu - \partial^\nu \tilde{\phi}^\mu = \partial^\mu(\phi^\nu + \partial^\nu \varphi) - \partial^\nu(\phi^\mu + \partial^\mu \varphi)$$

$$= \underbrace{\partial^\mu \phi^\nu - \partial^\nu \phi^\mu}_{F^{\mu\nu}} + \underbrace{\partial^\mu \partial^\nu \varphi - \partial^\nu \partial^\mu \varphi}_{0} = F^{\mu\nu}$$

also $F^{\mu\nu}$ eichinvariant !

Homogene Maxwellgln.

(1) $\underline{\nabla} \cdot \underline{B} = \partial_1 B^1 + \partial_2 B^2 + \partial_3 B^3 = 0$

$$\Rightarrow \partial_1 F^{32} + \partial_2 F^{13} + \partial_3 F^{21} = 0$$

mit $\partial_1 = -\partial^1, \ldots, \quad F^{32} = -F^{23}, \ldots$ folgt:

$$\boxed{\partial^1 F^{23} + \partial^2 F^{31} + \partial^3 F^{12} = 0} \qquad \text{zykl. } (1\,2\,3)$$

1 2 3   2 3 1   3 1 2

(2) $\underline{\nabla} \times \underline{E} + \frac{\partial}{\partial t} \underline{B} = 0$

1. Komp.: $\partial_2 E^3 - \partial_3 E^2 + \frac{\partial}{\partial t} B^1 = 0$

$$\Rightarrow \partial_2 F^{30} - \partial_3 F^{20} + \partial_0 F^{32} = 0$$

mit $\partial_0 = \partial^0, \ \partial_2 = -\partial^2, \ \partial_3 = -\partial^3, \quad F^{32} = -F^{23}, \ F^{20} = -F^{02}$:

$$\boxed{\partial^0 F^{23} + \partial^2 F^{30} + \partial^3 F^{02} = 0} \qquad \text{zykl. } (0\,2\,3)$$

zykl. Permutation $1 \to 2 \to 3 \to 1$ und $F^{\mu\nu} = -F^{\nu\mu}$

$$\boxed{\begin{array}{l} \partial^0 F^{13} + \partial^3 F^{01} + \partial^1 F^{30} = 0 \\ \partial^0 F^{12} + \partial^1 F^{20} + \partial^2 F^{01} = 0 \end{array}} \qquad \begin{array}{l} \text{zykl. } (0\,1\,3) \\ \text{zykl. } (0\,1\,2) \end{array}$$

Zusammenfassung der homog. Maxwellgln.:

$$\boxed{\epsilon_{\mu\nu\lambda\kappa} \partial^\nu F^{\lambda\kappa} = 0} \quad \text{oder} \quad \boxed{\epsilon^{\mu\nu\lambda\kappa} \partial_\nu F_{\lambda\kappa} = 0} \quad \text{„4-Rotation"}$$

mit

$$\epsilon^{\mu\nu\lambda\kappa} = \begin{cases} 1 & \text{wenn } (\mu\nu\lambda\kappa) = \text{gerade Permutation von } (0123) \\ -1 & \text{wenn } (\mu\nu\lambda\kappa) = \text{ungerade Perm. v. } (0123) \\ 0 & \text{sonst} \end{cases}$$

(4-dim. Levi-Civita-Tensor)

3-dim. Levi-Civita-Tensor $\epsilon_{ikl}\partial^k a^l = (\underline{\nabla} \times \underline{a})_i$

<u>Bemerkung</u>: (i) $\epsilon^{\mu\nu\lambda\kappa}$ ist vollständig antisymmetrisch

(ii) $\epsilon^{\mu\nu\lambda\kappa}$ transformiert sich unter Lorentz-Trafo:

$$\epsilon'^{\mu\nu\lambda\kappa} = U^\mu{}_\pi U^\nu{}_\rho U^\lambda{}_\sigma U^\kappa{}_\tau \epsilon^{\pi\rho\sigma\tau}$$

$$= \begin{vmatrix} U^\mu{}_0 & U^\mu{}_1 & U^\mu{}_2 & U^\mu{}_3 \\ U^\nu{}_0 & U^\nu{}_1 & U^\nu{}_2 & U^\nu{}_3 \\ U^\lambda{}_0 & U^\lambda{}_1 & U^\lambda{}_2 & U^\lambda{}_3 \\ U^\kappa{}_0 & U^\kappa{}_1 & U^\kappa{}_2 & U^\kappa{}_3 \end{vmatrix} = \underbrace{(\det U)}_{\pm 1} \epsilon^{\mu\nu\lambda\kappa}$$

Damit $\epsilon'^{\mu\nu\lambda\kappa} = \epsilon^{\mu\nu\lambda\kappa}$ ist, muss vereinbart werden, dass

$$\epsilon'^{\mu\nu\lambda\kappa} = \underbrace{(\det U)}_{\pm 1} U^\mu{}_\pi U^\nu{}_\rho U^\lambda{}_\sigma U^\kappa{}_\tau \epsilon^{\pi\rho\sigma\tau}$$

Damit ist $\epsilon^{\mu\nu\lambda\kappa}$ ein <u>Pseudotensor</u>

(Verallgemeinerung des 3-dim. Falles ϵ^{ikl} mit $(\underline{\nabla} \times \underline{a})_i = \epsilon^{ikl}\partial_k a_l$ Pseudovektor)