4er-Potential

$$\boxed{\begin{aligned}\phi^0 &:= \phi \\ \phi^i &:= cA^i\end{aligned}} \qquad (i=1,2,3)$$

Felder $\underline{E}$, $\underline{B}$

$$\underline{E} = -\underline{\nabla}\phi - \frac{\partial}{\partial t}\underline{A}$$

$$\Rightarrow E^i = -\partial_i\phi - \frac{1}{c}\frac{\partial}{\partial t}cA^i = -\partial_i\phi^0 - \partial_0\phi^i = \partial^i\phi^0 - \partial^0\phi^i$$

$$(i=1,2,3)$$

$$\underline{B} = \underline{\nabla}\times\underline{A}$$

$$\Rightarrow cB^1 = \partial_2 cA^3 - \partial_3 cA^2 = \partial_2\phi^3 - \partial_3\phi^2 = \partial^3\phi^2 - \partial^2\phi^3$$

zyklische Vertauschung: $\quad cB^2 = \partial^1\phi^3 - \partial^3\phi^1$

$$cB^3 = \partial^2\phi^1 - \partial^1\phi^2$$

Zusammenfassung antisymm. Feldtensor

$$\boxed{F^{\mu\nu} := \partial^\mu\phi^\nu - \partial^\nu\phi^\mu}, \quad F^{\mu\nu} = -F^{\nu\mu} \quad (\mu,\nu = 0,1,2,3)$$

Wegen der Antisymmetrie hat $F^{\mu\nu}$ nur 6 unabhängige Komponenten:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -cB^3 & cB^2 \\ E^2 & cB^3 & 0 & -cB^1 \\ E^3 & -cB^2 & cB^1 & 0 \end{pmatrix}$$

Lorentz-Trafo der Felder

Als Tensor 2. Stufe transformiert sich $F^{\mu\nu}$ wie

$$F'^{\mu\nu} = U^\mu{}_\lambda U^\nu{}_\kappa F^{\lambda\kappa}, \qquad U^\mu{}_\nu = \begin{pmatrix} \gamma & -\beta\gamma & & 0 \\ -\beta\gamma & \gamma & & \\ & 0 & & 1 \end{pmatrix}$$

Im einzelnen

$$E'^1 = F'^{10} = U^1{}_\lambda U^0{}_\kappa F^{\lambda\kappa} = -\beta\gamma U^0{}_\kappa F^{0\kappa} + \gamma U^0{}_\kappa F^{1\kappa}$$
$$= (\beta\gamma)^2 F^{01} + (\gamma)^2 F^{10}$$
$$= \underbrace{\gamma^2(1-\beta^2)}_{1} F^{10} = E^1$$

$$E'^2 = F'^{20} = U^2{}_\lambda U^0{}_\kappa F^{\lambda\kappa} = U^0{}_\kappa F^{2\kappa} = \gamma F^{20} - \beta\gamma F^{21}$$
$$= \gamma(E^2 - vB^3)$$

$$E'^3 = F'^{30} = U^3{}_\lambda U^0{}_\kappa F^{\lambda\kappa} = U^0{}_\kappa F^{3\kappa} = \gamma F^{30} - \beta\gamma F^{31}$$
$$= \gamma(E^3 + vB^2)$$

$$B'^1 = \frac{1}{c} F'^{32} = \frac{1}{c} U^3{}_\lambda U^2{}_\kappa F^{\lambda\kappa} = \frac{1}{c} F^{32} = B^1$$

$$B'^2 = \frac{1}{c} F'^{13} = \frac{1}{c} U^1{}_\lambda U^3{}_\kappa F^{\lambda\kappa} = \frac{1}{c} U^1{}_\lambda F^{\lambda 3} = -\frac{\beta\gamma}{c} F^{03} + \frac{\gamma}{c} F^{13}$$
$$= \gamma\left(B^2 + \frac{v}{c^2} E^3\right)$$

$$B'^3 = \gamma\left(B^3 - \frac{v}{c^2} E^2\right)$$

Zusammenfassung:

$$\boxed{\begin{array}{ll} E'^1 = E^1 & B'^1 = B^1 \\ E'^2 = \gamma(E^2 - vB^3) & B'^2 = \gamma\left(B^2 + \frac{v}{c^2}E^3\right) \\ E'^3 = \gamma(E^3 + vB^2) & B'^3 = \gamma\left(B^3 - \frac{v}{c^2}E^2\right) \end{array}}$$

Elektr. u. magn. Felder werden beim Übergang zwischen verschiedenen Inertialsystemen in einander transformiert.

Umeichung

$$\tilde{\phi}^\mu = \phi^\mu + \partial^\mu \varphi$$

$$\Rightarrow \tilde{F}^{\mu\nu} = \partial^\mu \tilde{\phi}^\nu - \partial^\nu \tilde{\phi}^\mu = \partial^\mu(\phi^\nu + \partial^\nu \varphi) - \partial^\nu(\phi^\mu + \partial^\mu \varphi)$$

$$= \underbrace{\partial^\mu \phi^\nu - \partial^\nu \phi^\mu}_{F^{\mu\nu}} + \underbrace{\partial^\mu \partial^\nu \varphi - \partial^\nu \partial^\mu \varphi}_{0}$$

$$= F^{\mu\nu}$$

also $F^{\mu\nu}$ eichinvariant !

Homogene Maxwellgln

(1) $\underline{\nabla} \cdot \underline{B} = \partial_1 B^1 + \partial_2 B^2 + \partial_3 B^3 = 0$

$$\Rightarrow \partial_1 F^{32} + \partial_2 F^{13} + \partial_3 F^{21} = 0$$

mit $\partial_1 = -\partial^1, \ldots,\ F^{32} = -F^{23}, \ldots$ folgt:

$$\boxed{\partial^1 F^{23} + \partial^2 F^{31} + \partial^3 F^{12} = 0} \quad \text{zykl. } (1\,2\,3)$$

$1\,2\,3 \quad 2\,3\,1 \quad 3\,1\,2$

(2) $\underline{\nabla} \times \underline{E} + \frac{\partial}{\partial t} \underline{B} = 0$

1. Komp.: $\partial_2 E^3 - \partial_3 E^2 + \frac{\partial}{\partial t} B^1 = 0$

$$\Rightarrow \partial_2 F^{30} - \partial_3 F^{20} + \partial_0 F^{32} = 0$$

mit $\partial_0 = \partial^0,\ \partial_2 = -\partial^2,\ \partial_3 = -\partial^3,\ F^{32} = -F^{23},\ F^{20} = -F^{02}$:

$$\boxed{\partial^0 F^{23} + \partial^2 F^{30} + \partial^3 F^{02} = 0} \quad \text{zykl. } (0\,2\,3)$$

zykl. Permut. $1 \to 2 \to 3 \to 1$ und $F^{\mu\nu} = -F^{\nu\mu}$

$$\boxed{\begin{aligned} &\partial^0 F^{13} + \partial^3 F^{01} + \partial^1 F^{30} = 0 \\ &\partial^0 F^{12} + \partial^1 F^{20} + \partial^2 F^{01} = 0 \end{aligned}} \quad \begin{aligned} &\text{zykl. } (0\,1\,3) \\ &\text{zykl. } (0\,1\,2) \end{aligned}$$

Zusammenfassung der homog. Maxwellgln.:

$$\boxed{\epsilon_{\mu\nu\lambda\kappa}\partial^\nu F^{\lambda\kappa} = 0} \quad \text{oder} \quad \boxed{\epsilon^{\mu\nu\lambda\kappa}\partial_\nu F_{\lambda\kappa} = 0}$$

„<u>4-Rotation</u>"

mit

$$\epsilon^{\mu\nu\lambda\kappa} = \begin{cases} 1 & \text{wenn } (\mu\nu\lambda\kappa) = \text{gerade Permutation von } (0123) \\ -1 & \text{wenn } (\mu\nu\lambda\kappa) = \text{ungerade} \quad \text{"} \\ 0 & \text{sonst} \end{cases}$$

(4-dim. Levi-Civita-Tensor)

3-dim. Levi-Civita-Tensor $\epsilon_{ikl}\partial^k a^l = (\underline{\nabla} \times \underline{a})_i$

<u>Bemerkung</u>: (i) $\epsilon^{\mu\nu\lambda\kappa}$ ist vollständig antisymmetrisch

(ii) $\epsilon^{\mu\nu\lambda\kappa}$ transformiert sich unter Lorentz-Trafo:

$$\epsilon'^{\mu\nu\lambda\kappa} = U^\mu{}_\pi U^\nu{}_\rho U^\lambda{}_\sigma U^\kappa{}_\tau \epsilon^{\pi\rho\sigma\tau}$$

$$= \begin{vmatrix} U^\mu{}_0 & U^\mu{}_1 & U^\mu{}_2 & U^\mu{}_3 \\ U^\nu{}_0 & U^\nu{}_1 & U^\nu{}_2 & U^\nu{}_3 \\ U^\lambda{}_0 & U^\lambda{}_1 & U^\lambda{}_2 & U^\lambda{}_3 \\ U^\kappa{}_0 & U^\kappa{}_1 & U^\kappa{}_2 & U^\kappa{}_3 \end{vmatrix} = \underbrace{(\det U)}_{\pm 1} \epsilon^{\mu\nu\lambda\kappa}$$

Damit $\epsilon'^{\mu\nu\lambda\kappa} = \epsilon^{\mu\nu\lambda\kappa}$ ist, muss vereinbart werden, dass

$$\epsilon'^{\mu\nu\lambda\kappa} = \underbrace{(\det U)}_{\pm 1} U^\mu{}_\pi U^\nu{}_\rho U^\lambda{}_\sigma U^\kappa{}_\tau \epsilon^{\pi\rho\sigma\tau}$$

Damit ist $\epsilon^{\mu\nu\lambda\kappa}$ ein <u>Pseudotensor</u>

(Verallgemeinerung des 3-dim. Falles ϵ^{ikl} mit

$(\underline{\nabla} \times \underline{a})_i = \epsilon^{ikl} \partial_k a_l$)

Pseudovektor = axialer Vektor : behält bei Punktspiegelung seine Richtung bei

polarer Vektor $\longleftrightarrow$