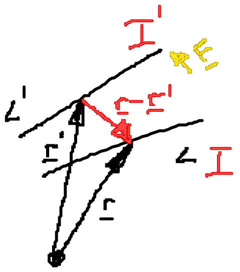
Nachtrag: Kraft zwischen 2 stromdurchflossenen Leitern



$$\underline{F} = -\frac{\mu_0}{4\pi} I I' \int_{L'}\int_{L} (d\underline{r}\, d\underline{r}') \frac{\underline{r}-\underline{r}'}{|\underline{r}-\underline{r}'|^3}$$

Kraft von L' auf L

parallele Ströme → Anziehung

antiparallele Ströme → Abstoßung

2.4. Magnetische Multipole

Ausgangspunkt: $\underline{A}(\underline{r}) = \frac{\mu_0}{4\pi} \int_{\mathbb{R}^3} d^3r' \frac{\underline{j}(\underline{r}')}{|\underline{r}-\underline{r}'|}$ [Coulomb Eichung $\operatorname{div}\underline{A}=0$]

mit Randbed. $\underline{A}(\underline{r}) \to 0$ für $\underline{r} \to \infty$

$r' \ll r$

$$\left[\frac{1}{r} \sum_{\ell=0}^{\infty} \left(\frac{r'}{r}\right)^{\ell} P_\ell(\xi) \right]$$

$$\frac{1}{|\underline{r}-\underline{r}'|} = \frac{1}{r} + \frac{(\underline{r}\cdot\underline{r}')}{r^3} + \dots$$

$$\Rightarrow \underline{A}(\underline{r}) = \underbrace{\frac{\mu_0}{4\pi r} \int_{\mathbb{R}^3} d^3r'\, \underline{j}(\underline{r}')}_{1} + \overbrace{\frac{\mu_0}{4\pi r^3} \int_{\mathbb{R}^3} d^3r'\, (\underline{r}\cdot\underline{r}')\, \underline{j}(\underline{r}')}^{2} + \dots$$

Monopol Term:

Mit $\nabla_{r'}\cdot\left(x_k'\,\underline{j}(\underline{r}')\right) = x_k'\underbrace{\left(\nabla_{r'}\cdot\underline{j}\right)}_{0} + \underbrace{\underline{j}\cdot\left(\nabla_{\underline{r}'}x_k'\right)}_{j_l\delta_{kl}} = j_k$

folgt: $\int d^3r'\,j_k(\underline{r}') = \int_{\mathbb{R}^3} d^3r'\,\nabla_{r'}\left[x_k'\underline{j}\right] = \int_{S(\infty)} d\underline{f}\cdot x_k'\underline{j} = \underline{\underline{0}}$

Monopol-Term verschwindet!

Dipol-Term:

$[b(ac)-c(ab)]$

Mit $\left(\underline{r}'\times\underline{j}\right)\times\underline{r} = (\underline{r}\cdot\underline{r}')\underline{j} - (\underline{r}\cdot\underline{j})\cdot\underline{r}'$

$= 2(\underline{r}\underline{r}')\underline{j} - \left[(\underline{r}\underline{r}')\underline{j} + (\underline{r}\cdot\underline{j})\underline{r}'\right]$

und $\nabla_{r'}\cdot\left\{x_k'(\underline{r}\cdot\underline{r}')\underline{j}\right\} = \left[(\underline{r}\cdot\underline{r}')j_k + x_k'(\underline{r}\cdot\underline{j}) + x_k'(\underline{r}\cdot\underline{r}')\underbrace{\nabla\underline{j}}_{0}\right]$

folgt $\underbrace{\int_{\mathbb{R}^3} d^3r'\,\nabla_{r'}\cdot\left\{x_k'(\underline{r}\cdot\underline{r}')\underline{j}\right\}}_{\underbrace{\int_{S_\infty} d\underline{f}'\left\{x_k'(\underline{r}\underline{r}')\underline{j}\right\}}_{0 \text{ da } \underline{j}\xrightarrow{r'\to\infty}0}} = \int d^3r'\left[(\underline{r}\underline{r}')j_k + (\underline{r}\underline{j})x_k'\right] \overset{!}{=} 0$

Also [...] gibt keinen Beitrag

$$\Rightarrow \underline{A}(\underline{r}) = \frac{\mu_0}{4\pi r^3} \frac{1}{2} \int_{\mathbb{R}^3} \underbrace{d^3r' \left(\underline{r}' \times \underline{j}(\underline{r}')\right)}_{} \times \underline{r}$$

Dipolterm

magnetisches Dipolmoment

$$\underline{m} := \frac{1}{2} \int d^3r' \, \underline{r}' \times \underline{j}(\underline{r}')$$

$$\boxed{\underline{A}(\underline{r}) = \frac{\mu_0}{4\pi r^3} \underline{m} \times \underline{r}}$$

Elektrostatik analog $\phi(\underline{r}) = \frac{1}{4\pi\varepsilon_0 r^3} \underline{p} \cdot \underline{r}$

$$\underline{p} := \int_{\mathbb{R}^3} d^3r' \, \underline{r}' \rho(\underline{r}')$$

$$\underline{E}(\underline{r}) = \frac{1}{4\pi\varepsilon_0} \frac{1}{r^5} \left[3(\underline{p}\cdot\underline{r})\underline{r} - r^2 \underline{p} \right]$$

Magnet. Induktion des Dipolmomentes $\underline{m}$:

$$\underline{B}(\underline{r}) = \text{rot}\left[\frac{\mu_0}{4\pi r^3} \underline{m} \times \underline{r}\right] = \underline{\frac{\mu_0}{4\pi} \frac{1}{r^5} \left[3(\underline{m}\cdot\underline{r})\underline{r} - r^2 \underline{m} \right]}$$

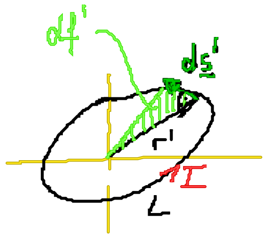
wegen:
$$\text{rot}(\underline{v}_1 \times \underline{v}_2) = (\underline{v}_2 \cdot \nabla)\underline{v}_1 - (\underline{v}_1 \cdot \nabla)\underline{v}_2 + \underline{v}_1(\nabla \cdot \underline{v}_2) - \underline{v}_2(\nabla \cdot \underline{v}_1)$$

$$\underline{v}_1 = \frac{\underline{m}}{r^3}, \quad \text{div}\, \underline{v}_1 = -\frac{3\,\underline{m}\cdot\underline{r}}{r^5}$$

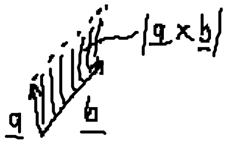
$$\underline{v}_2 = \underline{r}, \quad \text{div}\, \underline{v}_2 = 3$$

$$(\underline{v}_2 \cdot \nabla)\underline{v}_1 = -3\underline{m}\frac{r^2}{r^5}, \quad (\underline{v}_1 \cdot \nabla)\underline{v}_2 = \frac{\underline{m}}{r^3}$$

Beispiel: (i) ebene Leiterschleife L



$$d\underline{f}' = \frac{1}{2}\,\underline{r}' \times d\underline{s}' \qquad (*)$$



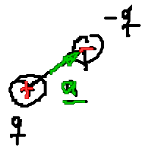
$$d^3r'\,\underline{j}(\underline{r}') = d\underline{s}'\; I$$

Strom durch L

$$\Rightarrow \underline{m} = \frac{1}{2}\oint_L d^3r'\; \underline{r}' \times \underline{j}(\underline{r}') = \frac{I}{2}\oint_L \underline{r}' \times d\underline{s}' \overset{(*)}{=} I\int_F d\underline{f} = \underline{\underline{I\cdot F\cdot \underline{n}}}$$

$\underline{n}$: Normale auf der von L eingeschl. Fläche

also Ringstrom → magn. Dipolmoment $\underline{m}$

[analog  $\underline{p} = \underline{a}\cdot q$]

(ii) Bewegte Ladungen

N Teilchen mit Masse m_i und Ladungen q_i

spezifische Ladung $\frac{q_i}{m_i} = \frac{q}{m} = \text{const.}$

$$\rho(\underline{r}) = \sum_i q_i\, \delta(\underline{r} - \underline{r}_i)$$

$$\underline{j}(\underline{r}) = \sum_i q_i\, \underline{v}_i\, \delta(\underline{r} - \underline{r}_i) \qquad \text{mit Geschwindigkeit } \underline{v}_i = \frac{d\underline{r}_i}{dt}$$

magn. Dipolmoment:

$$\underline{m} = \frac{1}{2}\int d^3r'\; \underline{r}' \times \underline{j}(\underline{r}') = \frac{1}{2}\sum_i q_i \int d^3r'\; \underline{r}' \times \underline{v}_i\, \delta(\underline{r}' - \underline{r}_i)$$

$$= \frac{1}{2} \sum_i q_i (\underline{r}_i \times \underline{v}_i) = \frac{1}{2} \sum_i \underbrace{\frac{q_i}{m_i}}_{q/m} m_i (\underline{r}_i \times \underline{v}_i)$$

$$= \frac{q}{2m} \underbrace{\sum_i m_i (\underline{r}_i \times \underline{v}_i)}_{\underline{L}} = \frac{q}{2m} \underline{L}$$

$\underline{L}$ Bahndrehimpuls

$$\boxed{\underline{m} = \frac{q}{2m} \underline{L}}$$ gilt auch für starre Körper

(aber nicht für Spin eines Elektrons

$\underline{m} = g \frac{e}{2m} \underline{S}$ mit $g \approx 2$

siehe QM)

Kraft auf Stromverteilung $\underline{j}(\underline{r}') = \rho(\underline{r}') \, \underline{v}(\underline{r}')$

im Feld einer magn. Induktion $\underline{B}(\underline{r}')$ (extern):

$$\underline{F} = \int_{\mathbb{R}^3} d^3r' \, \underline{j}(\underline{r}') \times \underline{B}(\underline{r}')$$ Lorentz Kraft

Taylorentwicklung: (Probedipol im geg. $\underline{B}$)

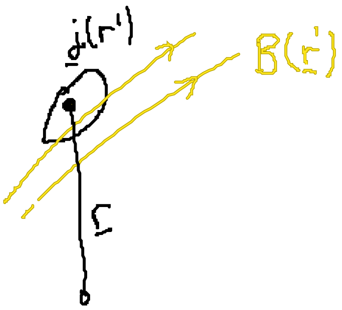
$$\underline{B}(\underline{r}') = \underline{B}(\underline{r}) + [(\underline{r}' - \underline{r}) \cdot \nabla_r] \, \underline{B}(\underline{r}) + \ldots$$

$$\Rightarrow \underline{F} = \underbrace{\left[\int d^3r' \, \underline{j}(\underline{r}') \right]}_{0} \times \underline{B}(\underline{r}) + \int d^3r' \, \underline{j}(\underline{r}') \times [(\underline{r}' - \underline{r}) \cdot \nabla_r] \, \underline{B}(\underline{r}) + \ldots$$

keine Monopole

$$= \int d^3r' \, \underline{j}(\underline{r}') \times \underbrace{\left[(\underline{r}' \cdot \nabla_r)\, \underline{B}(\underline{r}) \right]}_{\nabla_r(\underline{r}' \cdot \underline{B}(\underline{r})) - \underline{r}' \times \underbrace{(\nabla_r \times \underline{B}(\underline{r}))}_{0}} - \underbrace{\int d^3r' \, \underline{j}(\underline{r}')}_{0} \times \left[(\underline{r} \cdot \nabla_r)\, \underline{B}(\underline{r}) \right] + \ldots$$

(externes Feld soll keine Stromwirbel im Bereich $\underline{j}(\underline{r}')$ haben)



$$= \int d^3r' \, \underbrace{\underline{j}(\underline{r}') \times \nabla_r (\underline{r}' \cdot \underline{B}(\underline{r}))}_{-\nabla_r \times [(\underline{r}' \underline{B}) \underline{j}(\underline{r}')] + (\underline{r}' \underline{B}) \underbrace{(\nabla_r \times \underline{j}(\underline{r}'))}_{0}}$$

$$= - \nabla_r \times (\underline{m} \times \underline{B}(\underline{r}))$$

$$= (\underline{m} \cdot \nabla_r)\, \underline{B}(\underline{r}) = - \nabla_r \left(- \underline{m} \cdot \underline{B}(\underline{r}) \right)$$

$$\Rightarrow \boxed{V = - \underline{m} \cdot \underline{B}(\underline{r})}$$

Potenzielle Energie eines Dipols im Magnetfeld.

[analog : $V = - \underline{p} \, \underline{E}(\underline{r})$ für elektrischen Dipol im el. Feld]