Anmeldung unter
tuport.sap.tu-berlin.de
vor 28.6.19

Nabla-Operator:

$$\boxed{\underline{\nabla} = \underline{e}_i \frac{1}{\left|\frac{\partial \underline{r}}{\partial x_i}\right|} \frac{\partial}{\partial x_i}} \qquad (6.23)$$

kartesische Koordinaten:

$$\boxed{\underline{\nabla} = \underline{e}_x \frac{\partial}{\partial x} + \underline{e}_y \frac{\partial}{\partial y} + \underline{e}_z \frac{\partial}{\partial z}} \qquad (6.24)$$

6.5 Divergenz

Divergenz eines Vektorfeldes

$$\boxed{\operatorname{div} \underline{a}(\underline{r}) = \underline{\nabla} \cdot \underline{a}(\underline{r})} \qquad (6.34)$$

... Quellfeld von $\underline{a}(\underline{r})$

kartes. Koord.:

$$\underline{\nabla} \cdot \underline{a} = \frac{\partial}{\partial x} a_x + \frac{\partial}{\partial y} a_y + \frac{\partial}{\partial z} a_z \qquad (6.35)$$

Bsp: $\underline{a}(\underline{r}) = \underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \longrightarrow \underline{\nabla} \cdot \underline{r} = 3$

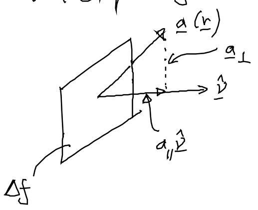
• Deutung: $\underline{\nabla} \cdot \underline{a}(\underline{r})$ identifiziert lokal Quellen & Senken von Vektorfeldern = „Flüssen" (6.40)

Bsp: $\underline{a}(\underline{r}) = \underline{v}(\underline{r})$ ... Geschw.feld einer Flüssigkeit

aber auch $\underline{E}(\underline{r})$, nicht $\underline{B}(\underline{r})$, da $\operatorname{div} \underline{B}(\underline{r}) = 0$
„keine magnet. Monopole"

Motivation:

(1) Fluß durch Fläche mit Normale $\hat{\underline{\nu}}$ $(|\hat{\underline{\nu}}| = 1)$ und Größe Δf

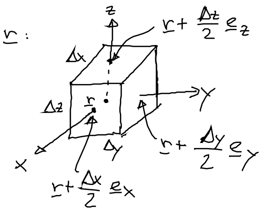


$a_\parallel \Delta f = (\underline{a} \cdot \hat{\underline{\nu}}) \Delta f$ ... tritt durch die Fläche

$\underline{a}_\perp \perp \hat{\underline{\nu}}$ nicht!

insbesondere: $\hat{\underline{\nu}} = \pm \underline{e}_x \longrightarrow a_\parallel = \pm a_x$

(2) Fluß aus kleinem Volumenelement $\Delta V = \Delta x \, \Delta y \, \Delta z$ um $\underline{r}$:



Aufgabe: Summe aller Flüsse aus ΔV hinaus!

Konvention: $\boxed{\hat{\underline{\nu}} \text{ zeigt immer aus } \Delta V \text{ hinaus}}$ (6.41)

Grund: $\underline{a} \cdot \hat{\underline{\nu}} > 0$ für Quelle

hier: $q_x^{\pm} = \overset{\hat{\underline{\nu}} = \underline{e}_x}{\underset{\hat{\underline{\nu}} = -\underline{e}_x}{\pm}} a_x\left(\underline{r} \pm \frac{\Delta x}{2} \underline{e}_x\right) \overset{\text{Taylor}}{=} \pm\left[a_x(\underline{r}) \pm \frac{\Delta x}{2} \frac{\partial}{\partial x} a_x\right]$

$$\left[f(\underline{r} + \Delta \underline{r}) = f(\underline{r}) + \underbrace{\Delta r_i \frac{\partial}{\partial x_i} f(\underline{r})}_{df}\right]$$

$q_y^{\pm} = \pm a_y\left(\underline{r} \pm \frac{\Delta y}{2} \underline{e}_y\right) = \pm\left[a_y(\underline{r}) \pm \frac{\Delta y}{2} \frac{\partial}{\partial y} a_y\right]$

$q_z^{\pm} = \ldots = \pm\left[a_z(\underline{r}) \pm \frac{\Delta z}{2} \frac{\partial}{\partial z} a_z\right]$

$\longrightarrow$ Fluß aus ΔV:

$$\begin{aligned}
q(\underline{r}) \, \Delta V &= \left(q_x^+ + q_x^-\right) \Delta y \, \Delta z \\
&\quad + \left(q_y^+ + q_y^-\right) \Delta x \, \Delta z + \ldots \\
&= \left\{\left[\cancel{a_x(\underline{r})} + \frac{\Delta x}{2} \frac{\partial}{\partial x} a_x\right] - \left[\cancel{a_x(\underline{r})} - \frac{\Delta x}{2} \frac{\partial}{\partial x} a_x\right]\right\} \Delta y \, \Delta z \\
&\quad + \ldots \Delta x \, \Delta z + \ldots \Delta x \, \Delta y \\
&= \left(\frac{\partial}{\partial x} a_x + \frac{\partial}{\partial y} a_y + \frac{\partial}{\partial z} a_z\right) \underbrace{\Delta x \, \Delta y \, \Delta z}_{\Delta V}
\end{aligned}$$

$\longrightarrow$ $\boxed{\begin{array}{l} q(\underline{r}) \, \Delta V \text{ mit } q(\underline{r}) = \operatorname{div} \underline{a}(\underline{r}) = \underline{\nabla} \cdot \underline{a}(\underline{r}) \\ \text{mißt Fluß aus } \Delta V \text{ heraus} \end{array}}$ (6.43)

Fälle: (i) $q(\underline{r}) = 0$ ... „was rein fließt, fließt auch raus"

(ii) > 0 ... „es fließt mehr raus als rein"

$\hat{=}$ Quelle

(iii) < 0 ... Senke

Bsp: $\underline{v}(\underline{r}) = v_0 \, \underline{r} \longrightarrow \operatorname{div} \underline{v} = 3 v_0$ (6.44)

$\longrightarrow$ überall Quellen!

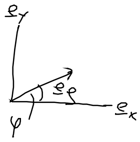
• Regeln: (1) $\underline{\nabla} \cdot (\underline{a} + \underline{b}) = \underline{\nabla} \cdot \underline{a} + \underline{\nabla} \cdot \underline{b}$

(2) $\underline{\nabla} \cdot [f(\underline{r}) \underline{a}] = f(\underline{r}) \, \underline{\nabla} \cdot \underline{a} + \underline{a} \cdot \underline{\nabla} f(\underline{r})$

Beweis in: kartes. Koordinatensystem ($\rightarrow$ Übungen)

• Zylinder Koordinaten:

$$\underline{\nabla}\cdot\underline{a}(\underline{r}) = \left(\underline{e}_\rho \frac{\partial}{\partial\rho} + \underline{e}_\varphi \frac{1}{\underline{\rho}}\frac{\partial}{\partial\varphi} + \underline{e}_z\frac{\partial}{\partial z}\right)\cdot\left(a_\rho\, \underline{e}_{\underline{\rho}} + a_\varphi\, \underline{e}_\varphi + a_z\, \underline{e}_z\right)$$



Achtung: (i) $\frac{\partial}{\partial\varphi}\underline{e}_\rho = \frac{\partial}{\partial\varphi}\begin{pmatrix}\cos\varphi\\ \sin\varphi\\ 0\end{pmatrix} = \begin{pmatrix}-\sin\varphi\\ \cos\varphi\\ 0\end{pmatrix} = \underline{e}_\varphi$,

$$\longrightarrow \frac{1}{\rho}a_\rho$$

(ii) $\frac{\partial}{\partial\varphi}\underline{e}_\varphi = -\underline{e}_\rho$, aber $\underline{e}_\varphi\cdot\underline{e}_\rho = 0$!

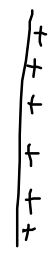
(iii) $\frac{\partial}{\partial x_i}\underline{e}_j = 0$ ... sonst

(6.46)

(6.46) und $\underline{e}_i\cdot\underline{e}_j = \delta_{ij}$

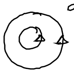
$$\longrightarrow \boxed{\begin{aligned}\underline{\nabla}\cdot\underline{a}(\underline{r}) &= \frac{\partial}{\partial\rho}a_\rho + \frac{1}{\rho}a_\rho + \frac{1}{\rho}\frac{\partial}{\partial\varphi}a_\varphi + \frac{\partial}{\partial z}a_z\\ &= \frac{1}{\rho}\frac{\partial(\rho a_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial}{\partial\varphi}a_\varphi + \frac{\partial}{\partial z}a_z\end{aligned}} \quad (6.47)$$

Bsp: (1) $\underline{E}(\underline{r}) \sim \underbrace{-\frac{1}{\rho}}_{a_\rho}\underline{e}_\rho$ ... Feld eines geladenen Drahtes



$\longrightarrow \operatorname{div}\underline{E}(\underline{r}) \sim \frac{1}{\rho^2} - \frac{1}{\rho^2} = 0,\ \rho \neq 0$

(2) $\underline{v}(\underline{r}) = \underbrace{\omega\rho}_{a_\varphi}\underline{e}_\varphi \longrightarrow \operatorname{div}\underline{v} = \frac{1}{\rho}\frac{\partial\omega\rho}{\partial\varphi} = 0$



• Kugel Koordinaten: $\underline{a} = a_r\,\underline{e}_r + a_\vartheta\,\underline{e}_\vartheta + a_\varphi\,\underline{e}_\varphi$

$$\boxed{\begin{aligned}\underline{\nabla}\cdot\underline{a}(\underline{r}) &= \frac{\partial a_r}{\partial r} + \frac{2}{r}a_r + \frac{1}{r}\frac{\partial a_\vartheta}{\partial\vartheta} + \frac{1}{r\tan\vartheta}a_\vartheta + \frac{1}{r\sin\vartheta}\frac{\partial a_\varphi}{\partial\varphi}\\ &= \frac{1}{r^2}\frac{\partial(r^2 a_r)}{\partial r} + \frac{1}{r\sin\vartheta}\frac{\partial(\sin\vartheta\, a_\vartheta)}{\partial\vartheta} + \frac{1}{r\sin\vartheta}\frac{\partial a_\varphi}{\partial\varphi}\end{aligned}} \quad (6.48)$$

Beweis: „Üben!"

Bsp: $\underline{a}(\underline{r}) = \underline{r} = \underbrace{r}_{a_r}\underline{e}_r = \frac{\partial r}{\partial r} + \frac{2}{r}r = 3!$ [vgl. (6.36)]

6.6 Rotation

• **Def:** $\boxed{\begin{array}{l}\text{Rotation eines Vektorfeldes } \underline{a}(\underline{r}) \\ \text{rot}\, \underline{a}(\underline{r}) = \underline{\nabla} \times \underline{a}(\underline{r}) \\ \ldots \text{ Wirbelfeld von } \underline{a}(\underline{r})\end{array}}$ (6.49)

• kartesische Koordinaten:

$$\underline{\nabla} = \underline{e}_i \frac{\partial}{\partial x_i}, \quad \underline{a}(\underline{r}) = a_i \underline{e}_i \quad , x_i, \quad i = x, y, z$$

$$\longrightarrow \quad \underline{\nabla} \times \underline{a}(\underline{r}) = \left(\underline{e}_i \frac{\partial}{\partial x_i}\right) \times \left(a_j(\underline{r})\, \underline{e}_j\right)$$

mit $\underline{e}_i \times \underline{e}_j = \varepsilon_{ijk}\, \underline{e}_k$ (2.10)

$$\left[\frac{\partial}{\partial x_i} \underline{e}_j = 0\right] \quad = \frac{\partial}{\partial x_i} a_j(r)\, \varepsilon_{ijk}\, \underline{e}_k$$

$\varepsilon_{ijk} = \varepsilon_{kij}$
$i \to j$
$j \to k$
$k \to i$

$$\boxed{\begin{aligned} \underline{\nabla} \times \underline{a}(\underline{r}) &= \varepsilon_{ijk} \left(\frac{\partial}{\partial x_i} a_j\right) \underline{e}_k \quad (6.50) \\ [\underline{\nabla} \times \underline{a}]_x &= \frac{\partial}{\partial y} a_z - \frac{\partial}{\partial z} a_y \\ [\underline{\nabla} \times \underline{a}]_y &= \frac{\partial}{\partial z} a_x - \frac{\partial}{\partial x} a_z \\ [\underline{\nabla} \times \underline{a}]_z &= \frac{\partial}{\partial x} a_y - \frac{\partial}{\partial y} a_x \end{aligned}}\left.\vphantom{\begin{aligned}a\\a\\a\end{aligned}}\right\} (6.51)$$

• **Deutung I:** $\boxed{\begin{array}{l}\text{rot}\, \underline{a}(\underline{r}) = \underline{\nabla} \times \underline{a}(\underline{r}) \text{ identifiziert} \\ \text{lokal Wirbel von Vektorfeldern}\end{array}}$ (6.52)

Bsp. 1: $\underline{v} = \underline{\omega} \times \underline{r}$ (6.53) [s. Kap. 6.2]

... Prototyp eines Wirbels im Geschw. feld

$\underline{\omega} = \frac{1}{2}\, \text{rot}\, \underline{v}$ ... Wirbelstärke (6.54)

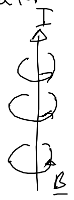
Beweis: $\text{rot}\, \underline{v} = \underline{\nabla} \times (\underline{\omega} \times \underline{r})$!!

Hilfsformel: $\underline{a} \times (\underline{b} \times \underline{c}) = \underline{b}(\underline{a}\cdot\underline{c}) - \underline{c}(\underline{a}\cdot\underline{b})$
& $\underline{\nabla}$ $\underline{r}$

$$= \underline{\omega}\, \underbrace{(\underline{\nabla}\cdot\underline{r})}_{3} - \underbrace{\underline{\omega}\cdot\underline{\nabla}\, \underline{r}}_{\omega_i \frac{\partial}{\partial x_i}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}\omega_x\\\omega_y\\\omega_z\end{pmatrix} = \underline{\omega}}$$

$$= 3\underline{\omega} - \underline{\omega} = 2\underline{\omega} \quad \text{qed}$$

Bsp 2: Strom-Leiter



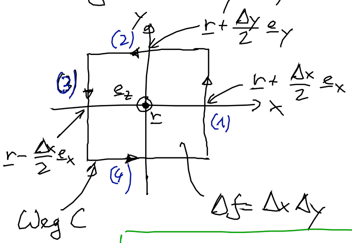
$\text{rot}\, \underline{B} \sim I$ (6.55)
(Maxwell-Gl. !!)

• **Deutung II:**

Ziel: Verdeutlichung Maß für Wirbel von $\underline{a}(\underline{r})$ um Flächenelement Δf

(i) Orientierte Fläche = Flächenelement Δf mit Normalenvektor $\hat{\underline{\nu}}$ ($|\hat{\underline{\nu}}|=1$) $\longrightarrow \Delta \underline{f} = \Delta f \, \hat{\underline{\nu}}$

hier: $\Delta f = \Delta x \Delta y$, $\hat{\underline{\nu}} = \hat{\underline{e}}_z$

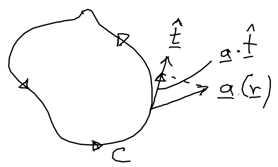


(ii) Konvention: $\boxed{\text{Wähle Weg } C \text{ um } \Delta f \text{ mit Rechter-Hand-Regel}}$

hier: $\hat{\underline{\nu}} = \underline{e}_z$, $C = (1) \longrightarrow (2) \rightarrow (3) \longrightarrow (4)$

(iii) Zirkulation/Wirbel:

$\boxed{\begin{array}{l}\text{Summiere Tangentialkomp. } \underline{a}\cdot\hat{\underline{t}} \\ \text{von } \underline{a}(\underline{r}) \text{ entlang } C \\ \hat{\underline{t}} \ldots \text{ Tangentialvektor an } C \text{, orientiert wie } C\end{array}}$



hier:

$$\Big[\underbrace{a_y\big(\underline{r}+\tfrac{\Delta x}{2}\underline{e}_x\big)}_{\underline{a}\cdot\underline{e}_y \ (1)} - \underbrace{a_y\big(\underline{r}-\tfrac{\Delta x}{2}\underline{e}_x\big)}_{\underline{a}\cdot(-\underline{e}_y) \ (3)}\Big]\Delta y$$

$$+ \Big[\underbrace{a_x\big(\underline{r}-\tfrac{\Delta y}{2}\underline{e}_y\big)}_{\underline{a}\cdot\underline{e}_x \ (4)} - \underbrace{a_x\big(\underline{r}+\tfrac{\Delta y}{2}\underline{e}_y\big)}_{\underline{a}\cdot(-\underline{e}_x) \ (2)}\Big]\Delta x$$

$$\overset{\text{Taylor}}{=} \Big[\cancel{a_y(\underline{r})} + \frac{\Delta x}{2}\frac{\partial}{\partial x}a_y - \cancel{a_y(\underline{r})} + \frac{\Delta x}{2}\frac{\partial}{\partial x}a_y\Big]\Delta y$$

$$+ \Big[\cancel{a_x(\underline{r})} - \frac{\Delta y}{2}\frac{\partial}{\partial y}a_x - \cancel{a_x(\underline{r})} - \frac{\Delta y}{2}\frac{\partial}{\partial y}a_x\Big]\Delta x$$

$$= \Big(\frac{\partial}{\partial x}a_y - \frac{\partial}{\partial y}a_x\Big)\Delta x \Delta y = \underbrace{(\text{rot}\,\underline{a})_z}_{(\text{rot}\,\underline{a})\cdot\underline{e}_z} \underbrace{\Delta x \Delta y}_{\Delta f}$$

beliebige Orientierung von $\Delta \underline{f}$ $\rightarrow$

$$\boxed{\begin{array}{c}\text{Verwirbelung/Zirkulation um } \Delta f \\ (\text{rot}\,\underline{a})\cdot\Delta\underline{f} = \text{rot}\,\underline{a}\cdot\hat{\underline{\nu}}\,\Delta f\end{array}} \quad (6.57)$$