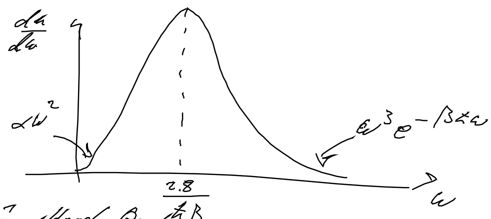
WdH

- Planck'sches Strahlungsgesetz

$$\frac{du}{d\omega} = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{e^{\beta\hbar\omega}-1}$$



Ableitung: $\varepsilon(k) = \hbar c \cdot k$ $\hat{=}$ ultrarel. Quantengas

$p_{\text{nichtrel.}} = \frac{2}{3} \cdot \frac{u}{V}$ $\quad p_{\text{rel}} = \frac{1}{3} \cdot \frac{u}{V}$

$$\frac{u}{V} = 2 \cdot \frac{1}{2\pi^2} \int_0^\infty \frac{\hbar c\, k^3 dk}{e^{\beta\hbar c k}-1} \qquad \frac{N}{V} = 2 \cdot \frac{1}{2\pi^2} \int \frac{k^2 dk}{e^{\beta\hbar c k}-1}$$

2 Polarisationen $\quad \mu_{\text{phot}} = 0$

- Quanten-Ising-Modell

$\alpha \in \{x, y, z\}$

$$\sigma_i^\alpha = \mathbb{1}_1 \otimes \ldots \otimes \mathbb{1}_{i-1} \otimes \sigma^\alpha \otimes \mathbb{1}_{i+1} \otimes \ldots \otimes \mathbb{1}_N$$

$$H_{\text{Ising}} = \underbrace{-g \sum_{i=1}^N \sigma_i^x}_{H_0} \underbrace{- J \sum_{i=1}^N \sigma_i^z \sigma_{i+1}^z}_{H_1} \qquad \sigma_{N+1}^z = \sigma_1^z$$

$$[H_0, H_1] \neq 0$$

V, W seien Hilberträume
$\{|v_i\rangle\}$ $\{|w_i\rangle\}$
$V \otimes W$ ist auch ein HR
$\{|v_i\rangle \otimes |w_j\rangle\}$ ist eine Basis in $V \otimes W$

$$C = \sum_\tau A_\alpha \otimes B_\tau$$

wirkt nur auf W

A_α wirkt nur auf V

$$|\psi\rangle = \sum_{ij} C_{ij} |v_i\rangle \otimes |w_j\rangle$$

$$C|\psi\rangle = \sum_{ij} C_{ij} \sum_\tau \left(A_\alpha |v_i\rangle\right) \otimes \left(B_\tau |w_j\rangle\right)$$

$$H_{\text{Ising}} = \sum_k \varepsilon_k \left(\gamma_k^+ \gamma_k - \tfrac{1}{2}\right)$$

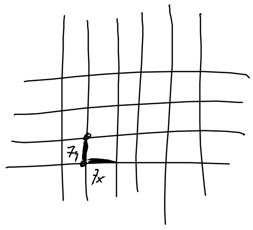
$$\varepsilon_k = 2\sqrt{g^2 + J^2 - 2gJ \cdot \cos\left(\tfrac{2\pi k}{N}\right)}$$

$$\rightarrow E_0 = \sum_k \varepsilon_k \left(-\tfrac{1}{2}\right) \rightsquigarrow \frac{E_0}{N} \rightarrow \int_{-1/2}^{+1/2} \varepsilon(k)\, dk$$

kleinste Anregung $k = 1/2$ $\rightsquigarrow E_1 = E_0 + 2 \cdot \varepsilon_{1/2}$

Modell verhält sich nichtanalytisch bei $g = J$ & $N \to \infty$
- auch bei $T = 0$ "Quantenphasenübergang"
- nichtanalyt. Verhalten ist in Observablen sichtbar.

Verbindung zum 2d Ising-Modell



$$Z_C = \operatorname{Tr}\{e^{-\beta H}\} = \operatorname{Tr}\{e^{+\beta g \sum \sigma_i^x + \beta J \sum_i \sigma_i^z \sigma_{i+1}^z}\}$$

$$= \sum_{\underline{z}} \langle z_1 \ldots z_N | e^{-\beta H} | z_1 \ldots z_N\rangle$$

$\sigma^z|0\rangle = +1|0\rangle$
$\sigma^z|1\rangle = -1|1\rangle$

$$\sigma_i^z |z_1 \ldots z_N\rangle = (-1)^{z_i} |z_1 \ldots z_N\rangle$$

$$|z_1\rangle \otimes |z_2\rangle \otimes \ldots \otimes |z_N\rangle = |\underline{z}\rangle = |z_1 \ldots z_N\rangle$$

Trotter-Formel

$$e^{A+B} = \lim_{L\to\infty}\left(e^{A/L}\, e^{B/L}\right)^L \qquad \sum_i |z_i\rangle\langle z_i| = \mathbb{1}$$

$$Z_C = \sum_{z_1} \lim_{L\to\infty} \langle z_1 | \underbrace{\left(e^{\beta H_0/L}\, e^{\beta H_1/L}\right)\cdot\ldots\cdot\left(e^{-\beta H_0/L}\, e^{-\beta H_1/L}\right)}_{L \text{ mal}} | z_1\rangle$$

- $\langle z_\tau | e^{+\frac{\beta J}{L}\sum_i \sigma^z_i \sigma^z_{i+1}} | z_{\bar\tau}\rangle = e^{\frac{\beta J}{L}\sum_i \sigma^i_\tau \sigma^{i+1}_\tau} \cdot \delta_{z_\tau z_{\bar\tau}}$

- $\langle z_\tau | e^{\frac{\beta g}{L}\sum_i \sigma^x_i} | z_{\bar\tau}\rangle = \ldots \qquad [\sigma^x_i, \sigma^x_j] = 0$

$$e^{\frac{\beta g}{L}\sigma^x} = \sum_{n=0}^\infty \left(\frac{\beta g}{L}\right)^n \frac{1}{n!}(\sigma^x)^n = \sum_{n=0}^\infty \left(\frac{\beta g}{L}\right)^{2n}\frac{1}{(2n)!}\cdot\mathbb{1} + \sum_{n=0}^\infty\left(\frac{\beta g}{L}\right)^{2n+1}\frac{1}{(2n+1)!}\cdot\sigma^x$$

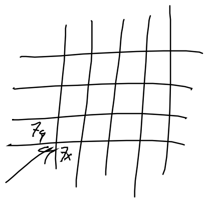
$$= \cosh\left(\frac{\beta g}{L}\right)\cdot\mathbb{1} + \sinh\left(\frac{\beta g}{L}\right)\cdot\sigma^x = \begin{pmatrix}\cosh\left(\frac{\beta g}{L}\right) & \sinh\left(\frac{\beta g}{L}\right)\\ \sinh\left(\frac{\beta g}{L}\right) & \cosh\left(\frac{\beta g}{L}\right)\end{pmatrix}$$

$$\langle z^i_\tau | e^{\frac{\beta g}{L}\sigma^x_i} | z^i_{\bar\tau}\rangle = \cosh\left(\frac{\beta g}{L}\right)\cdot\delta_{z^i_\tau z^i_{\bar\tau}} + \sinh\left(\frac{\beta g}{L}\right)\left[1-\delta_{z^i_\tau z^i_{\bar\tau}}\right]$$
$$= \Lambda\cdot e^{\gamma\, \sigma^i_\tau \sigma^i_{\bar\tau}}$$

$$Z_C \longrightarrow \sum_{\sigma^i_\tau\in\{-1,+1\}} \Lambda^{N\cdot L} \exp\left\{\underbrace{\frac{\beta J}{L}}_{\bar\beta\cdot J_x}\sum_{i\tau}\sigma^i_\tau\sigma^{i+1}_\tau + \underbrace{\gamma}_{\bar\beta\cdot J_y}\sum_{i\tau}\sigma^i_\tau\sigma^i_{\tau+1}\right\}$$

$$J = \frac{L}{\beta}\cdot\bar\beta\cdot J_x$$

$$g = \frac{L}{\beta}\cdot\operatorname{arcoth}\left(e^{2\bar\beta\cdot J_y}\right)$$



$x \hat= i$
$y \hat= \tau$

krit. Punkt: $g = J \rightsquigarrow \coth(\bar\beta_{crit}\cdot J_x) = e^{2\bar\beta\cdot J_y}$

$J_x = J_y = J_{class}$

$$\beta_c\cdot J_{class} = \frac{1}{2}\ln\left(1+\sqrt{2}\right) \approx 0.441$$

Einschub: kollektive Spinmodelle (Drehimpulskopplung)

Wie beschreibt man Modelle mit kollektiven Spins

$$J^\alpha = \frac{1}{2}\sum_{n=1}^N \sigma^\alpha_n \qquad \alpha\in\{x,y,z\}$$

$$[\sigma^x_a, \sigma^y_b] = 2i\,\sigma^z_a\cdot\delta_{ab}$$

$$J^2 = (J^x)^2+(J^y)^2+(J^z)^2 \quad \Rightarrow \quad \boxed{[J^x, J^y] = i\cdot J^z}$$

suche Basis $|j, m\rangle$ in der J^2 & J^z diagonal sind $\Rightarrow [J^2, J^z] = 0$

Bsp.: $N=2$

$|00\rangle$

$|01\rangle$ $|10\rangle$ sind entartet

$|11\rangle$

$$|00\rangle \quad \frac{1}{\sqrt{2}}[|01\rangle + |10\rangle] \quad \frac{1}{\sqrt{2}}[|01\rangle - |10\rangle] \quad |11\rangle$$

$$[J^z, J^\pm] \stackrel{!}{=} \pm J^\pm \rightsquigarrow \boxed{J^\pm = J^x \pm i J^y}$$

Leiteroperatoren

$$\boxed{\begin{aligned} J^2 |j m\rangle &= j(j+1) |j m\rangle \\ J^z |j m\rangle &= m |j m\rangle \\ J^\pm |j m\rangle &= \underbrace{\sqrt{j(j+1) - m(m\pm 1)}}_{\text{Clebsch-Gordon-Koeff.}} |j, m\pm 1\rangle \end{aligned}}$$

$-j \le m \le +j$

$$j \in \begin{cases} 0, 1, \ldots \frac{N}{2} & \text{falls } N \text{ gerade} \\ \frac{1}{2}, \frac{3}{2}, \ldots \frac{N}{2} & \text{falls } N \text{ ungerade} \end{cases}$$

Bsp.: $N=2$

$|00\rangle \hat{=} |j=1, m=+1\rangle$

$\frac{1}{\sqrt{2}}[|01\rangle + |10\rangle] \hat{=} |j=1, m=0\rangle$ $\quad$ $\frac{1}{\sqrt{2}}[|01\rangle - |10\rangle] \hat{=} |j=0, m=0\rangle$

$|11\rangle \hat{=} |j=1, m=-1\rangle$

$z_1 = 1 \quad z_2 = 1$

$N=3$ $\quad$ $j = 1/2$

$|000\rangle \hat{=} |j=\frac{3}{2}, m=+\frac{3}{2}\rangle$

$\frac{1}{\sqrt{3}}[|100\rangle + |010\rangle + |001\rangle] \hat{=} |j=\frac{3}{2}, m=+\frac{1}{2}\rangle$

$\frac{1}{\sqrt{3}}[|110\rangle + |011\rangle + |101\rangle] \hat{=} |j=\frac{3}{2}, m=-\frac{1}{2}\rangle$

$|111\rangle \hat{=} |j=\frac{3}{2}, m=-\frac{3}{2}\rangle$

$J^+ \quad J^-$

$j = 3/2 \qquad j = 1/2 \qquad j = 1/2$

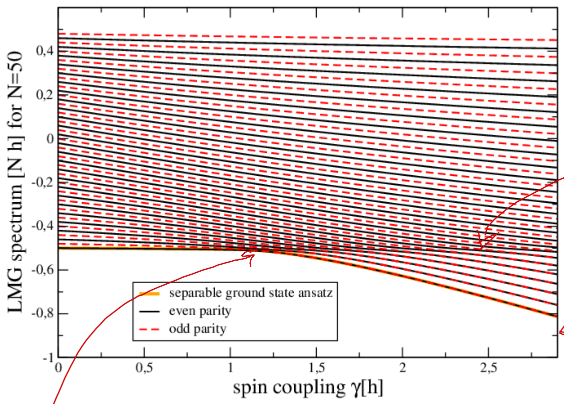
3.3.3. Lipkin-Meshkov-Glick-Modell

- historisch: Dynamik von Kernspins in ext. Feld

$$\hat{H} = -h\hat{J}^z - \frac{\gamma}{N}\left(\hat{J}^x\right)^2 \qquad \left[J^z, (J^x)^2\right] \neq 0$$

$$\Sigma = e^{i\pi\frac{N}{2} + i\pi\hat{J}^z} \qquad [\hat{H}, \hat{J}^2] = 0$$

$$\rightsquigarrow \hat{\Sigma}\hat{J}^x\hat{\Sigma} = -J^x \qquad [H, \Sigma] = 0$$



ESQPT

$\lim N \to \infty \quad \gamma = h$ QPT

Schrittweise Diagonalisierung

$$\boxed{\begin{aligned} J^+ &= \sqrt{N\cdot\mathbb{1} - a^\dagger a}\; a \\ J^- &= a^\dagger \sqrt{N\cdot\mathbb{1} - a^\dagger a} \\ J^z &= \frac{N}{2}\cdot\mathbb{1} - a^\dagger a \end{aligned}}$$

HP-Trafo

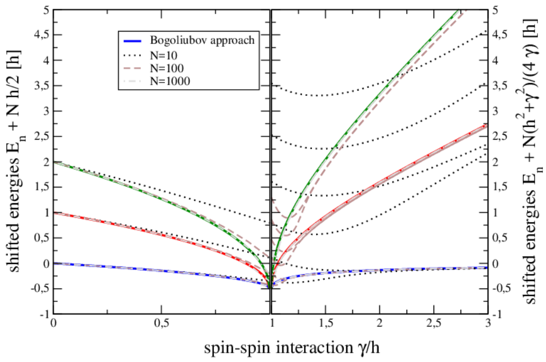
$$[J^x, J_y] = i J^z$$
$$\Longleftrightarrow [a, a^\dagger] = 1$$

- $N \to \infty$ (TD-Limes): Wurzel entwickeln
- $\hat{a} = \hat{b} + \bar{\alpha}$, $\bar{\alpha} \in \mathbb{C}$ $\qquad \Rightarrow [b, b^\dagger] = 1$
- $b^\dagger b,\ b b^\dagger,\ b^2,\ (b^\dagger)^2$

$$b = \cosh(\varphi)\, c + \sinh(\varphi)\, c^\dagger \qquad \Rightarrow [c, c^\dagger] = 1$$

$\varphi \in \mathbb{R}$

$$\Rightarrow \boxed{H = \pm E(h,\gamma)\, c^\dagger c + \alpha_1(h,\gamma) - \alpha_2(h,\gamma)\cdot N}$$



Bogoliubov approach
N=10
N=100
N=1000
shifted energies E_n + N h/2 [h]
shifted energies E_n + N(h^2+γ^2)/(4 γ) [h]
spin-spin interaction γ/h