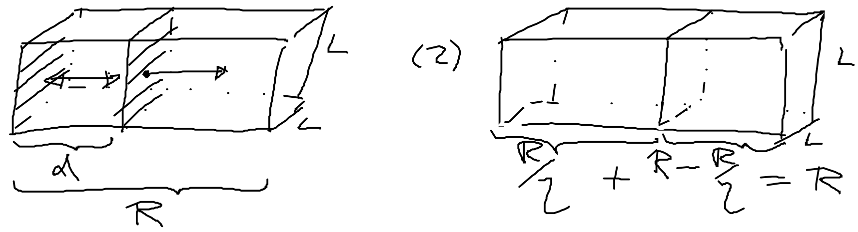
Casimir-Effekt: Rechnung über den Casimir-Kolben (oder Hubraum)

aus Greiner, Special topics in Quantum Mechanics
und William M.R. Simpson,
„Surprises in Theoretical Casimir Physics",
Springer (2015), published PhD-thesis

(1) (2)



$$U(d,L) = \frac{\hbar c}{2} \sum_{m,n,l=0}^{\infty} k_{lmn}(d,L,L)\, \Theta(n,m,l)\, \underbrace{e^{-\lambda k_{lmn}}}_{\text{cut-off Funktion}}$$

Physikalische Motivation: elektromagnetisch induzierte Transparenz
(bei hohen Frequenzen werden die Leiterplatten transparent für das Licht)

$$\Theta(n,m,l) = \begin{cases} 1, & n \vee m \vee l = 0 \\ 2, & n \wedge m \wedge l \neq 0 \end{cases}$$

$$U(d,L) = \frac{\hbar c}{2} \sum_{l=0}^{\infty} \sum_{\substack{m=0\\n=0}}^{\infty} \frac{\Delta m}{\Delta m} \frac{\Delta n}{\Delta n} \sqrt{\left(\frac{l\pi}{d}\right)^2 + \left(\frac{m\pi}{L}\right)^2 + \left(\frac{n\pi}{L}\right)^2}\, e^{-\lambda k_{lmn}}\, \Theta(n,m,l)$$

$\left(\Delta m = \frac{L}{\pi}\right)$ $\quad d \ll L$

$$= \frac{\hbar c L^2}{2\pi^2} \sum_{l=0}^{\infty} \int_0^{\infty} dm \int_0^{\infty} dn\, \underbrace{\sqrt{\left(\frac{l\pi}{d}\right)^2 + m^2 + n^2}}_{=: k_l(m,n)}\, e^{-\lambda k_l(m,n)}\, \Theta(l)$$

$\Theta(l) = \Theta(1)$ für $l > 0$

$$= \frac{\hbar c L^2}{2\pi^2} \Big\{ \underbrace{\int_0^{\infty} dm \int_0^{\infty} dn \sqrt{m^2+n^2}\, e^{-\lambda k_0(m,n)}\, \Theta(0)}_{u_0(\lambda)} + \Theta(1) \sum_{l=1}^{\infty} \int_0^{\infty} dm \int_0^{\infty} dn \sqrt{\left(\frac{l\pi}{d}\right)^2 + m^2 + n^2}\, e^{-\lambda k_l(m,n)} \Big\}$$

$u_0(\lambda)$ der ist unabhängig von d, L (Offset)

$$U(d,\lambda) = \frac{\Theta(\lambda)\hbar c L^2}{2\pi^2} \sum_{l=1}^{\infty} \int_0^{\infty} dm \int_0^{\infty} dn \sqrt{\left(\frac{l\pi}{d}\right)^2 + m^2 + n^2}\; e^{-\lambda R_l(m,n)}$$

$$= \frac{\Theta(\lambda)\hbar c L^2}{2\pi^2} \sum_{l=1}^{\infty} \int\int \left(\frac{l\pi}{d}\right) \sqrt{1 + m^2\left(\frac{d}{l\pi}\right)^2 + n^2\left(\frac{d}{l\pi}\right)^2}\; e^{-\lambda \frac{l\pi}{d}\sqrt{\dots}}$$

$$m = \frac{l\pi}{d}\sqrt{z}\,\cos\varphi$$
$$n = \frac{l\pi}{d}\sqrt{z}\,\sin\varphi$$

$$= \frac{\Theta(\lambda)\hbar c L^2}{2\pi^2} \sum \int_0^{\infty}\int_0^{\pi/2} dz\, d\varphi \underbrace{\left|\frac{\partial(m,n)}{\partial(z,\varphi)}\right|}_{\frac{l^2\pi^2}{d^2}\frac{1}{2}} \frac{l\pi}{d}\sqrt{1+z}\; e^{-\lambda\frac{l\pi}{d}\sqrt{1+z}}$$

$$= \frac{\Theta(\lambda)\hbar c L^2}{2\pi^2} \underline{\underline{\sum_{l=1}^{\infty}}} \left(\frac{l\pi}{d}\right)^3 \frac{\pi}{4} \int_0^{\infty} dz \sqrt{1+z}\; \underline{\underline{e^{-\lambda\frac{l\pi}{d}\sqrt{1+z}}}}$$

geometrische Reihe $\bar{\lambda} = \pi\lambda$

$$\frac{d^3}{d\bar{\lambda}^3} e^{-\bar{\lambda}\frac{l}{d}\sqrt{1+z}} = \left(-\frac{l}{d}\sqrt{1+z}\right)^3 e^{-\bar{\lambda}\frac{l}{d}\sqrt{\dots}}$$

$$= -\frac{\Theta(\lambda)\hbar c L^2}{8}\pi^2 \underline{\underline{\sum_{l=1}^{\infty}}} \int_0^{\infty} \frac{dz}{1+z} \left(\underline{\underline{\frac{d^3}{d\bar{\lambda}^3}}} e^{-\bar{\lambda}\frac{l}{d}\sqrt{1+z}}\right)$$

$$= -\frac{\Theta(\lambda)\hbar c L^2}{8}\pi^2 \frac{d^3}{d\bar{\lambda}^3} \int_0^{\infty} \frac{dz}{1+z} \underbrace{\sum_{l=1}^{\infty} e^{-\bar{\lambda}\frac{l}{d}\sqrt{1+z}}}$$

$$= \frac{1}{\overline{e^{\bar{\lambda}\frac{1}{\alpha}\sqrt{1+z}} - 1}}$$

$$= \frac{1}{1-q} - 1 = \frac{q}{1-q}$$

$$u = \sqrt{1+z} \qquad du = \frac{1}{2\sqrt{1+z}}\, dz$$

$$= -\frac{\Theta(1)\hbar c L^2 \pi^2}{4} \frac{d^3}{d\bar{\lambda}^3} \int_1^\infty du\, \frac{1}{u} \frac{1}{e^{\bar{\lambda}\frac{u}{\alpha}} - 1}$$

$$= \Theta(1) \frac{\hbar c L^2 \pi^2}{4} \frac{1}{\alpha} \frac{d^2}{d\bar{\lambda}^2} \int_1^\infty du\, \frac{e^{\bar{\lambda}\frac{u}{\alpha}}}{\left(e^{\bar{\lambda}\frac{u}{\alpha}} - 1\right)^2}$$

$$y = e^{\bar{\lambda}\frac{u}{\alpha}} \qquad dy = \frac{\bar{\lambda}}{\alpha} e^{\bar{\lambda}\frac{u}{\alpha}}\, du$$

$$y\Big|_{u=1}\; e^{\bar{\lambda}/\alpha} \qquad y\Big|_{u\to\infty} = \infty$$

$$= \Theta(1) \frac{\hbar c L^2 \pi^2}{4\alpha} \frac{d^2}{d\bar{\lambda}^2} \int_{e^{\bar{\lambda}/\alpha}}^\infty dy\, \frac{\alpha}{\bar{\lambda}} \frac{1}{(y-1)^2}$$

$$\bar{y} := y - 1 \qquad d\bar{y} = dy$$

$$= \Theta(1) \frac{\hbar c L^2 \pi^2}{4\alpha} \frac{d^2}{d\bar{\lambda}^2} \int_{e^{\bar{\lambda}/\alpha}-1}^\infty d\bar{y}\, \frac{\alpha}{\bar{\lambda}} \frac{1}{\bar{y}^2}$$

$$= \Theta(1)\,[\dots]\,\frac{d^2}{d\bar\lambda^2}\left\{\frac{d}{\bar\lambda}\,\frac{d}{\bar\lambda}\,\frac{(\bar\lambda/d)}{e^{\bar\lambda/d}-1}\right\}$$

Bernoulli-Formel $\frac{y}{e^y-1}=\sum_{n=0}^{\infty}\frac{B_n y^n}{n!}$

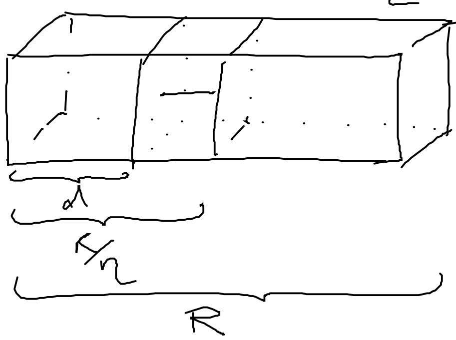
$\left[B_0=1,\ B_1=-\frac{1}{2},\ B_2=\frac{1}{6},\ \underline{\underline{B_3=0}},\ B_4=-\frac{1}{30}\right]$ special points

$$= \Theta(1)\,\underbrace{[\dots]\,\frac{d^2}{d\bar\lambda^2}\left\{\frac{1}{d}\sum_{n=0}^{\infty}\right.}\frac{B_n}{n!}\left(\frac{\bar\lambda}{d}\right)^{n-2}\Bigg\}$$

$$U(d,\lambda,L) = U_0(\lambda) + \Theta(1)\,\frac{\hbar c L^2\pi^2}{2}\,\frac{d^2}{d\bar\lambda^2}\left\{\frac{1}{d}\sum_{n=0}^{\infty}\frac{B_n}{n!}\left(\frac{\bar\lambda}{d}\right)^{n-2}\right\}$$

Berechnet wird die Energiedifferenz

$$\Delta U(d,\lambda,R) = \underline{U(d,\lambda)} + \underline{U(R-d,\lambda)} - \left[\underline{U\left(\tfrac{R}{2},\lambda\right)} + \underline{U\left(R-\tfrac{R}{2},\lambda\right)}\right]$$



$L^2=A\to\infty$

$R\to\infty$

$$C_\alpha := \Theta(1)\,\frac{\hbar c L^2\pi^2}{4}$$

$$\Delta U = U_0 + C_\alpha\,\frac{d^2}{d\bar\lambda^2}\left\{B_0\cancel{\frac{d}{\bar\lambda^2}} + B_1\frac{1}{\bar\lambda} + B_2/2d + B_3\frac{\bar\lambda}{d^2} + B_4\frac{\bar\lambda^2}{d^3} + \dots\right\}$$

$$+U_0 + C_\alpha \frac{d^2}{d\bar{\Lambda}^2}\left\{ B_0 \cancel{\frac{R-d}{\bar{\Lambda}^2}} + B_1 \frac{1}{\bar{\Lambda}} + B_2 \frac{1}{2(R-d)} + \dots \right\}$$

$$-U_0 - C_\alpha \frac{d^2}{d\bar{\Lambda}^2}\left\{ \cancel{\quad} \right\}$$

$$\underline{\underline{-U_0 - C_\alpha \frac{d^2}{d\bar{\Lambda}^2}}}\left\{ B_0 \cancel{\frac{R-\frac{R}{2}}{\bar{\Lambda}^2}} + B_1 \frac{1}{\bar{\Lambda}} + \frac{B_2}{2(R-\frac{R}{2})} + \frac{B_3}{6}\frac{\bar{\Lambda}}{(R-\frac{R}{2})} \right.$$

B_2 Terme, weil unabhängig von $\bar{\Lambda}^2$

$$\frac{\Delta U}{C_\alpha}(d,\bar{\Lambda},R) = \frac{B_4 \cdot 2 \cdot 1}{24}\left[\frac{1}{d^3} + \frac{1}{(R-d)^3} - \frac{1}{(\frac{R}{2})^3} - \frac{1}{(R-\frac{R}{2})^3}\right]$$

$$+ \frac{B_5}{5!}\cdot 3\cdot 2\left[\frac{\bar{\Lambda}}{d^4} + \frac{\bar{\Lambda}}{(R-d)^4} - \frac{\bar{\Lambda}}{(\frac{R}{2})^4} - \frac{\bar{\Lambda}}{(R-\frac{R}{2})^4}\right] + \frac{B_6}{6!}\dots$$

$$\lim_{\bar{\Lambda}\to 0} \frac{U}{C_\alpha}(d,\bar{\Lambda},R) = \frac{B_4}{12}\left[\frac{1}{d^3} + \frac{1}{(R-d)^3} - \frac{1}{(\frac{R}{2})^3} - \frac{1}{(R-\frac{R}{2})^3}\right]$$

$$\frac{U(d)}{C_\alpha} := \lim_{R\to\infty} \frac{U(d,R)}{C_\alpha} = \frac{B_4}{12}\frac{1}{d^3} = \frac{1}{-30}\frac{1}{12}\frac{1}{d^3}$$

$$U(d) = -C_\alpha \frac{1}{360}\frac{1}{d^3} = -\frac{\hbar c \pi^2}{720}\frac{L^2}{d^3}\underbrace{\frac{\Theta(1)}{2}}_{=1}$$

(Experiment)

damit Casimir-Effekt hergeleitet

- ohne Maclaurin-Euler
- ohne Zeta-Riemann
- ohne heikle Berücksichtigung der verschwindenden zweiten Polarisationsmode
- <u>und</u> unabhängig vom cut-off $\bar{\Lambda}$!!

- Milonni leitet sogar Ausdruck

$$H = \frac{\hbar c}{2} \sum_{m,n,p} \eta_{lmn}(d,L,L)\,\Theta(n,m,p)$$

über den klassischen Strahlungsdruck aus der Drehimpulserhaltung der Maxwellfelder her

- alles leitet Schwinger allein aus der Van-der-Waals WW der leitenden Platten her

(1.) Unschärferelation : Quantenmetrologie (Atomuhren, Sensing)

(2.) EPR-Paradoxon, Bellschen Ungleichungen: Quanteninformation (Kryptographie-Protokolle)

(3.) Aharanov-Bohm Effekt: Quantum Topologie (fraktionalen QHE, TI) [Strom-Normale-Definition]

(4.) Casimir-Effekt : Energie des Vakuums (Energiequelle ?) [Quantum sensing]

(5.) Hanbury Brown-Twiss Effekt : ....