English Summary

Information gain $K(P,P') = \sum_{i=1}^{N} P_i \ln \frac{P_i}{P'_i}$

generalized canonical distripution

Jaynes' Principle of unbiased guess:

maximize missing information

$$I(P) = \sum_{i=1}^{N} P_i \ln P_i = \text{Min}$$

under constraints $\sum_i P_i = 1$

$$\langle M^\nu \rangle_{\text{macro}} = \sum_i P_i M_i^\nu \qquad \nu = 1, \dots, m$$

micro-observables

$$\Rightarrow \boxed{P_i = \exp(\psi - \lambda_\nu M_i^\nu)}$$

normalization $\Rightarrow$ $\boxed{e^{-\psi} = \sum_i \exp(-\lambda_\nu M_i^\nu) \equiv Z}$ partition fct.

Legendre transform $I(M) = \psi(t) - Mt$, $M \equiv \frac{d\psi}{dt} \Rightarrow \frac{dI}{dM} = -t$

Shannon-Information:

$$I(P) = \sum_i P_i \ln P_i = \sum_i P_i (\psi - \lambda_\nu M_i^\nu) = \psi - \lambda_\nu \sum_i P_i M_i^\nu$$

$$\boxed{I = \psi(\lambda_1, \dots, \lambda_m) - \lambda_\nu \langle M^\nu \rangle}$$

Aus $\psi(\lambda_1, \dots, \lambda_m) = -\ln \sum_i \exp(-\lambda_\mu M_i^\mu)$ folgt

$$\frac{\partial}{\partial \lambda_\nu} \psi = - \frac{\sum_i (-M_i^\nu) \exp(-\lambda_\mu M_i^\mu)}{\underbrace{\sum_i \exp(-\lambda_\mu M_i^\mu)}_{e^{-\psi}}} = \sum_i M_i^\nu \underbrace{\exp(\psi - \lambda_\mu M_i^\mu)}_{P_i}$$

$$\boxed{\frac{\partial \psi}{\partial \lambda_\nu} = \langle M^\nu \rangle}$$

Damit können wir die Legendre-Trafo identifizieren:

$$\Psi(t) \longrightarrow \Psi(\lambda_1, \dots, \lambda_m) \qquad \text{Var. } \lambda_\nu$$

$$M \longrightarrow \langle M^\nu \rangle = \frac{\partial \Psi}{\partial \lambda_\nu} \qquad \text{neue Var. } \langle M^\nu \rangle$$

$$I(M) \longrightarrow I = \Psi - \lambda_\nu \langle M^\nu \rangle \qquad \text{Legendre-Trafo von } \Psi$$

Es folgt $\boxed{\frac{\partial I}{\partial \langle M^\nu \rangle} = -\lambda_\nu}$

wegen $\frac{\partial I}{\partial \langle M^\nu \rangle} = \cancel{\underbrace{\frac{\partial \Psi}{\partial \lambda_\mu}}_{\langle M^\mu \rangle} \frac{\partial \lambda_\mu}{\partial \langle M^\nu \rangle} - \frac{\partial \lambda_\mu}{\partial \langle M^\nu \rangle} \langle M^\mu \rangle} - \lambda_\nu$

Zusammengefasst:

$$\boxed{dI = -\lambda_\nu \, d\langle M^\nu \rangle}$$

(Thermodyn.: Gibbs'sche Fundamentalgl.)

Betrachte Variation

$$\langle M^\nu \rangle \longrightarrow \langle M^\nu \rangle + \delta \langle M^\nu \rangle$$

$$\lambda_\nu \longrightarrow \lambda_\nu + \delta \lambda_\nu$$

$$\Psi \longrightarrow \Psi + \delta \Psi$$

$$P_i \longrightarrow P_i + \delta P_i$$

Informationsgewinn:

$$K(P+\delta P, P) = \underbrace{\sum_i (P_i + \delta P_i) \ln (P_i + \delta P_i)}_{I(P+\delta P)} - \sum_i (P_i + \delta P_i) \ln P_i$$

$$= (\cancel{\Psi} + \delta\Psi) - (\cancel{\lambda_\nu} + \delta\lambda_\nu)(\langle M^\nu \rangle + \delta \langle M^\nu \rangle) - \underbrace{\sum_i (P_i + \delta P_i)(\Psi - \lambda_\nu M_i^\nu)}_{\cancel{\Psi} - \lambda_\nu \cancel{\underbrace{\sum_i (P_i + \delta P_i) M_i^\nu}_{\langle M^\nu \rangle + \delta \langle M^\nu \rangle}}}$$

$$= \delta\Psi - \delta\lambda_\nu (\langle M^\nu \rangle + \delta \langle M^\nu \rangle)$$

Entwicklung für kleine Var $\delta\lambda_\nu$:

$$\delta\Psi = \frac{\partial\Psi}{\partial\lambda_\nu}\delta\lambda_\nu + \frac{1}{2}\frac{\partial^2\Psi}{\partial\lambda_\nu\partial\lambda_\mu}\delta\lambda_\nu\delta\lambda_\mu + \ldots$$

$$\delta\langle M^\nu\rangle = \frac{\partial\langle M^\nu\rangle}{\partial\lambda_\mu}\delta\lambda_\mu + \ldots$$

$$\Rightarrow K(P+\delta P, P) = \underbrace{\left(\frac{\partial\Psi}{\partial\lambda_\nu} - \langle M^\nu\rangle\right)}_{0}\delta\lambda_\nu + \left[\frac{1}{2}\frac{\partial}{\partial\lambda_\mu}\underbrace{\left(\frac{\partial\Psi}{\partial\lambda_\nu}\right)}_{\langle M^\nu\rangle} - \frac{\partial\langle M^\nu\rangle}{\partial\lambda_\mu}\right]\delta\lambda_\nu\delta\lambda_\mu$$

$$= -\frac{1}{2}\frac{\partial\langle M^\nu\rangle}{\partial\lambda_\mu}\delta\lambda_\mu\delta\lambda_\nu \overset{!}{\geq} 0$$

Also $\frac{\partial\langle M^\nu\rangle}{\partial\lambda_\mu}\delta\lambda_\mu\delta\lambda_\nu \leq 0$ <u>negativ semidefinite Bilinearform</u> $\forall\,\delta\lambda_\mu$

Definiere <u>Suszeptibilitätsmatrix</u>

$$\chi^{\mu\nu} := \frac{\partial\langle M^\mu\rangle}{\partial\lambda_\nu} = \frac{\partial^2\Psi}{\partial\lambda_\nu\partial\lambda_\mu}$$ (Änderung von $\langle M^\nu\rangle$ bei Var. von λ_ν; $\delta\langle\underline{M}\rangle = \underline{\underline{\chi}}\,\delta\underline{\lambda}$)

$$\tilde{\chi}_{\sigma\lambda} := \frac{\partial\lambda_\sigma}{\partial\langle M^\lambda\rangle} = -\frac{\partial^2 I}{\partial\langle M^\nu\rangle\partial\langle M^\sigma\rangle}$$ ($\delta\underline{\lambda} = \underline{\underline{\tilde{\chi}}}\,\delta\langle\underline{M}\rangle$)

$$\boxed{\underline{\underline{\tilde{\chi}}} = \underline{\underline{\chi}}^{-1}}$$

Wegen
$$\underbrace{\frac{\partial}{\partial\lambda_\mu}\underbrace{\left(\frac{\partial\Psi}{\partial\lambda_\nu}\right)}_{\langle M^\nu\rangle}}_{\chi^{\nu\mu}} = \underbrace{\frac{\partial}{\partial\lambda_\nu}\underbrace{\left(\frac{\partial\Psi}{\partial\lambda_\mu}\right)}_{\langle M^\mu\rangle}}_{\chi^{\mu\nu}}$$

Symmetrische Matrix

Aus $K(P+\delta P, P) \geq 0$ folgt

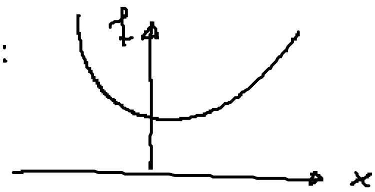
$$\boxed{\chi^{\mu\nu}\delta\lambda_\mu\delta\lambda_\nu = \delta\langle M^\nu\rangle\delta\lambda_\nu = \tilde{\chi}_{\nu\mu}\,\delta\langle M^\nu\rangle\delta\langle M^\mu\rangle \leq 0}$$

negativ-semidefinite quadrat. Form

$$\Rightarrow \chi^{\nu\nu} \leq 0 \quad , \quad \tilde{\chi}_{\nu\nu} \leq 0$$

<u>NB</u> : $\Rightarrow I(\langle M^\nu\rangle)$ und $-\psi(\lambda_\nu)$ <u>konvex</u> !

(konvexe Fkt. :



$f''(x) \geq 0$)

<u>Zusammenhang mit Korrelationsmatrix</u> :

$$Q^{\mu\nu} := \langle \Delta M^\mu \Delta M^\nu \rangle \qquad \text{Korrelationsmatrix}$$

$$= \langle M^\mu M^\nu \rangle_c \qquad \text{2. Kumulante}$$

$$= \left.\frac{\partial^2 \Gamma(\alpha)}{\partial\alpha_\mu \partial\alpha_\nu}\right|_{\alpha=0} \qquad \text{mit Kumulantenerzeugender}$$

$$\Gamma(\alpha) = \ln\langle \exp(\alpha_\nu M^\nu)\rangle$$

$$= \ln \sum_i P_i \exp(\alpha_\nu M_i^\nu)$$

$$= \ln \sum_i e^{\psi - (\lambda_\nu - \alpha_\nu) M_i^\nu}$$

$$= \underbrace{\ln\left[e^{\psi}\right]}_{=\psi(\lambda)} + \underbrace{\ln \sum_i \exp(-(\lambda_\nu - \alpha_\nu) M_i^\nu)}_{-\psi(\lambda-\alpha)}$$

$$(e^{-\psi} = Z)$$

$$\Gamma(\alpha) = \psi(\lambda) - \psi(\lambda - \alpha) :$$

$$Q^{\mu\nu} = -\left.\frac{\partial^2 \Phi(\lambda-\alpha)}{\partial\alpha_\mu \partial\alpha_\nu}\right|_{\alpha=0} = -\frac{\partial^2 \Phi(\lambda)}{\partial\lambda_\mu \partial\lambda_\nu} = -\chi^{\mu\nu}$$

Korr. Suszeptibilität

Also

$$\boxed{\langle \Delta M^\mu \Delta M^\nu \rangle = -\frac{\partial \langle M^\mu \rangle}{\partial \lambda_\nu} = -\frac{\partial \langle M^\nu \rangle}{\partial \lambda_\mu}}$$

Fluktuations-Dissipations-Theorem

zufällige Schwankung um Mittelwert

systematische Änderung der Mittelwerte

$$\Delta M^\mu = M^\mu - \langle M^\mu \rangle$$