English Summary:

Dissipative systems

$$\frac{dV_t}{dt} = \Lambda V_t \Rightarrow V(t) = V_0 e^{\Lambda t}$$ phase space volume shrinks

$$\Lambda = \mathrm{div}\,\underline{F} = \mathrm{tr} A = \sum_i \mathrm{Re}\,\lambda_i < 0$$ (asymptotically stable fixed point)

attractor A : closed, invariant, undecomposable, asymptotically stable set

attractor basin U_0 : $\phi_t(U_0) \xrightarrow{t\to\infty} A$

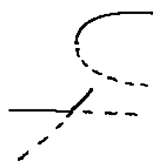
attractors : $d=1$ stable fixed points
$d=2$ " limit cycle
$d=3$ " torus
strange attrange

1.3 Bifurkationen

(A) Eigenwert-Null-Bifurkation

(A1) Sattel-Knoten-Bifurkation

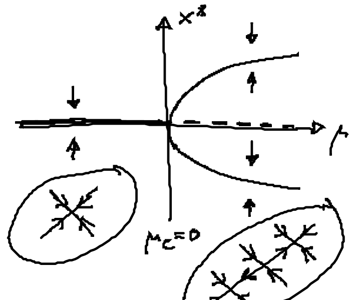
(A2) Transkrit. Bifurkation



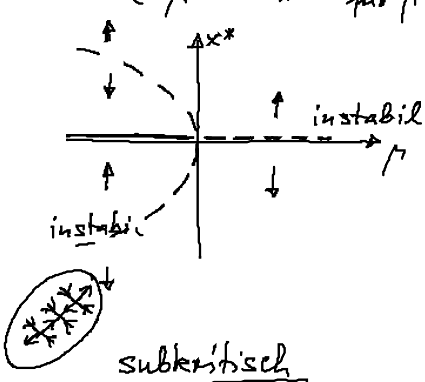
(A3) Stimmgabel-Bifurkation (pitchfork bifurcation)

$\boxed{\dot{x} = \mu x - x^3}$ $\qquad \delta\dot{x} = (\mu - 3x^{*2})\delta x$

$$x^* = \begin{cases} \pm\sqrt{\mu} & (\text{für } \mu \ge 0) \\ 0 & \end{cases} \qquad \lambda = \begin{cases} -2\mu & \text{stabil für } \mu > 0 \\ \mu & \text{" für } \mu < 0 \end{cases}$$



Superkritisch



subkritisch

$\boxed{\dot{x} = \mu x + x^3}$

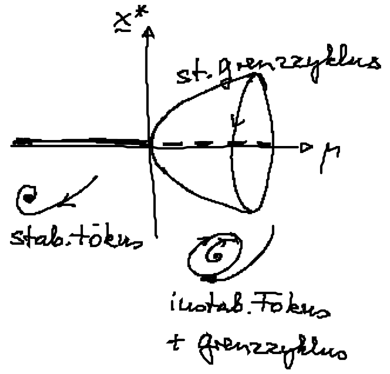
(B) Hopf-Bifurkation (Andronov-Hopf): superkritisch

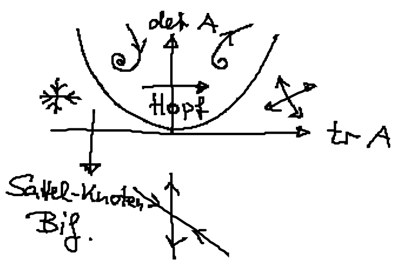
$\lambda = \lambda_0 \pm i\omega$ mit $\lambda_0 < 0 \longrightarrow \lambda_0 > 0$

stabiler Fokus $\longrightarrow$ instab. Fokus + stab. Grenzzyklus

$\underline{n=2}$: $\mathrm{tr}A < 0 \longrightarrow \mathrm{tr}A > 0$

(Vor.: $\det A > 0$)





min. $\boxed{n=2}$ nötig!

Hopf-Normalform: generische Taylorentwicklung in der Nähe der Hopf-Bif.
(2D-Zentrums-Mannigfaltigkeit)

$$\boxed{\dot z = \Big(\underbrace{\lambda + i\omega}_{\text{lin. Anteil}} \mp \underbrace{(1+i\gamma)|z|^2}_{\text{nichtlin. Anteil}}\Big) z} \qquad z = x + iy \in \mathbb{C}$$

$\Rightarrow$ Bifurkation eines Grenzzyklus
(limit cycle LC)
= period. Orbit

$\delta\dot z = (\lambda + i\omega)\,\delta z$ lin. um Fixpkt. $z^* = 0$

$$\left.\begin{array}{l}\text{Fixpkt. } z = 0\\ \text{Eigenwert } \lambda + i\omega\end{array}\right\}\quad \begin{array}{l}\lambda < 0: \text{ stab. Fokus}\\ \lambda > 0: \text{ instab. Fokus}\end{array}$$

$$\begin{pmatrix}\dot x\\ \dot y\end{pmatrix} = \begin{pmatrix}\lambda & \omega\\ -\omega & \lambda\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} \quad \Rightarrow \Lambda = \frac{\mathrm{tr}A}{2} \pm \sqrt{\left(\frac{\mathrm{tr}A}{2}\right)^2 - \det A}$$

$$= \lambda \pm \sqrt{\cancel{\lambda^2} - \cancel{\lambda^2} - \omega^2}$$

$$= \lambda \pm i\omega$$

Transform. auf Amplitude r und Phase φ:

$$z(t) = r(t)\, e^{i\varphi(t)} \quad \Rightarrow \quad \dot r e^{i\varphi} + i\dot\varphi\, r e^{i\varphi} = \left(\lambda + i\omega \mp (1+i\gamma) r^2\right) r e^{i\varphi}$$

$$\begin{array}{ll} \text{Re:} & \dot r = (\lambda \mp r^2)\, r \\ \text{Im:} & \dot\varphi = (\omega \mp \gamma r^2) \end{array}$$

Re: $\Rightarrow r = 0$ oder $r^2 = \pm\lambda \quad (\lambda \gtrless 0)$

Im: $\Longrightarrow \dot\varphi = \omega - \gamma\lambda$ Frequ.

$\Rightarrow \varphi = (\omega - \gamma\lambda)\, t$

Amplitud.-Phasen-Koppl.

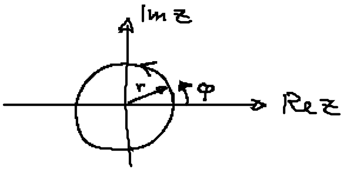
$\Downarrow$

Periode $T = \frac{2\pi}{\dot\varphi} = \frac{2\pi}{\omega - \gamma\lambda}$

Lös.: $r = 0$ (Fixpkt.)

$$z(t) = \sqrt{\pm\lambda}\; e^{i(\omega - \gamma\lambda)t} \quad \text{für } \lambda \gtrless 0$$

Stuart-Landau-Oszillator



Im Bifurk.pkt. ($\lambda = 0$)

Amplitude $r = \sqrt{\pm\lambda} \longrightarrow 0$

Frequenz $\omega \neq 0$

Lin. Stabilität des LC:

im allgemeinen: **Floquet-Theorie**

$\dot z = f(z)$, perio. Orbit $z^*(t) = z^*(t+T)$

$$\delta\dot z = Df\Big|_{z^*,t}\, \delta z \quad \text{mit } Df(t) = Df(t+T),$$

lin. ODE mit perio. Koeff.

Lösung $\delta z(t) = \sum_j c_j\, e^{\Lambda_j t}\, u_j(t)$ mit $u_j(t) = u_j(t+T)$

Floquet-Mode

Floquet-Exponenten $\Lambda_j \in \mathbb{C}$

$\Rightarrow \Lambda u + \dot u = Df\, u, \quad \delta z(t) = U(t)\, \delta z(0)$

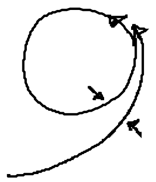
Floquet-Multiplikatoren $\mu = e^{\Lambda T}$ (stabil falls $|\mu| < 1$)

= Eigenwerte von $U(T)$

<u>hier</u>: analyt. Lösung möglich in r, φ:

$$\begin{pmatrix}\delta\dot r\\ \delta\dot\varphi\end{pmatrix} = \begin{pmatrix}\lambda \mp 3r^2 & 0\\ \mp 2\gamma r & 0\end{pmatrix}\Bigg|_{z^*(t)} \begin{pmatrix}\delta r\\ \delta\varphi\end{pmatrix} = \underbrace{\begin{pmatrix}-2\lambda & 0\\ \mp 2\gamma\sqrt{\pm\lambda} & 0\end{pmatrix}}_{A}\begin{pmatrix}\delta r\\ \delta\varphi\end{pmatrix}$$

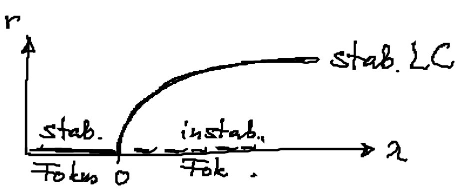
Floquet-Exponenten sind die Eigenwerte von A:



$$\Lambda^2 + 2\lambda\Lambda = 0$$

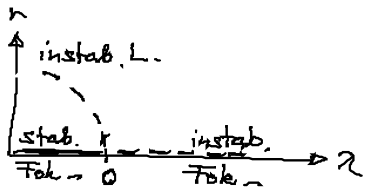
$$\Rightarrow \Lambda = \begin{cases} 0 & \text{Goldstone-Mode (long. Bewegung entlang LC)} \\ -2\lambda \lessgtr 0 \quad (\lambda \gtrless 0) & \text{(transversaler Floquet-Exp.)} \end{cases}$$

Bifurkationsdiagramm:



<u>superkrit. Hopf-Bif.</u>

$$\dot z = \left(\lambda + i\omega - (1 + i\gamma)|z|^2\right) z$$



<u>subkrit. Hopf-Bif.</u>

$$\dot z = \left(\lambda + i\omega + (1 + i\gamma)|z|^2\right) z$$