Wdh

- holomorphe Fkt $f(z) = u(x,y) + i\,v(x,y)$ $\qquad z = x+iy$

$$\frac{\partial u}{\partial x} = + \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = - \frac{\partial v}{\partial x}$$

$$\rightsquigarrow \left\{\partial_x^2 + \partial_y^2\right\} u(x,y) = 0$$

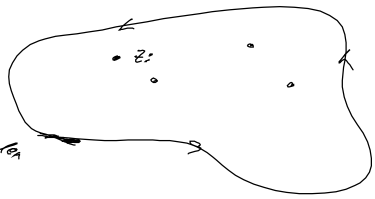
- Laurent-Reihe

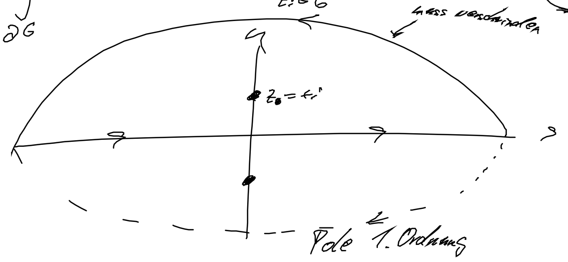
$$f(z) = \sum_{n=-\infty}^{\infty} a_n \cdot (z-z_0)^n$$

+ hebbare Sing. $a_{n<0} = 0 \longrightarrow \frac{\sin(z)}{z}$ bei $z=0$
+ Pol der Ordnung $(n>0)$: $a_{n<-n} = 0 \quad a_{-n} \neq 0 \rightarrow \frac{1}{(z-i)^3}$
+ essentielle Sing. $e^{1/z}$
+ $a_{-1} \longrightarrow$ Residuum von $f(z)$ an z_0

- Residuensatz

$$\oint_{\partial G} f(z)\,dz = 2\pi i \sum_{z_i \in G} \operatorname*{Res}_{z=z_i} f(z)$$





Pole 1. Ordnung

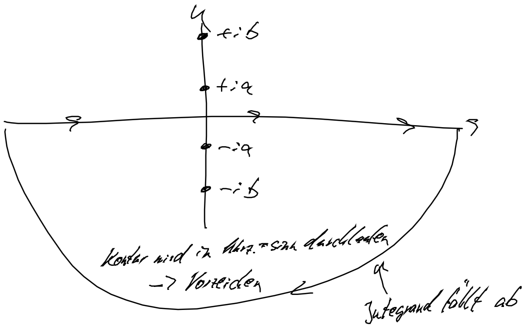
$$a_{-1} = \lim_{z\to z_0} (z-z_0)\cdot f(z)$$

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \oint \frac{dz}{(z-i)(z+i)} = 2\pi i \left.\frac{1}{z+i}\right|_{z=i} = \pi$$

$$F(k) = \int \frac{dx\; e^{-ikx}}{(x^2+a^2)(x^2+b^2)} = \int \frac{dz \quad e^{-ikz}}{\underline{(z+ia)}(z-ia)(z+ib)(z-ib)}$$

$$e^{-ik(x+iy)} = e^{-ikx}\, e^{+ky}$$

$a>0 \quad b>0 \qquad F(-k) = F(+k) \rightsquigarrow a, b, k > 0$



$$\operatorname*{Res}_{z=-ia} f(z) = \frac{e^{-ik(-ia)}}{(-2ia)(-ia+ib)(-ia-ib)}$$

$$F(k) = \frac{\pi}{b^2-a^2}\left[\frac{e^{-|ka|}}{|a|} - \frac{e^{-|kb|}}{|b|}\right]$$

- Hauptwert-Integrale $(a>0)$

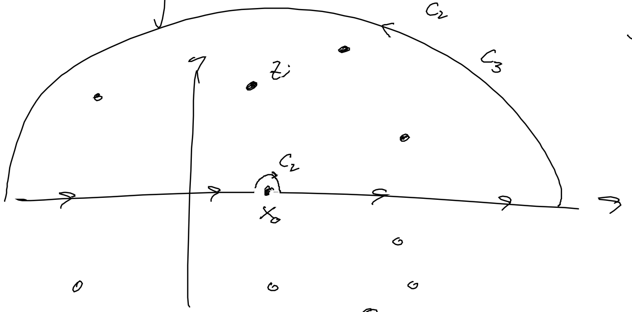
$$\mathcal{P}\int_{-a}^{+a} \frac{dx}{x} = 0 \qquad \mathcal{P}\int_a^b g(x)\,dx := \lim_{\varepsilon\to 0}\left[\int_a^{c-\varepsilon} g(x)\,dx + \int_{c+\varepsilon}^{b} g(x)\,dx\right] \qquad c \in (a,b)$$

Def. Hauptwert

Wenn $g(x) = \frac{f(x)}{x-x_0}$ $\quad f(x)$ holomorph $\forall x \in \mathbb{R}$

$$\mathcal{P}\int \frac{f(x)}{x-x_0}dx + \int_{C_2}\frac{f(z)}{z-x_0}dz + \underbrace{\int_{C_3}\frac{f(z)}{z-x_0}}_{\to 0} = 2\pi i \sum_{z_i:\,\mathrm{Im}(z_i)>0} \mathrm{Res}_{z=z_i}\frac{f(z)}{z-z_0}$$

falls $f(z) \propto \frac{1}{R^{1+\varepsilon}}$



$$\int_{C_2}\frac{f(z)}{z-x_0}dz = \lim_{R\to 0}\int_{\pi}^{0}\frac{f(x_0+R\cdot e^{i\varphi})}{\cancel{R\cdot e^{i\varphi}}}\, i\,\cancel{R\, e^{i\varphi}}\,d\varphi = f(x_0)\cdot(-\pi\cdot i)$$

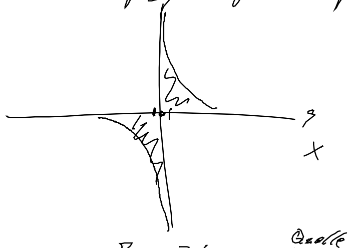
$$\boxed{\mathcal{P}\int\frac{f(x)}{x-x_0}dx = \pi i\cdot f(x_0) + 2\pi i\sum_{z_n:\,\mathrm{Im}(z_n)>0}\mathrm{Res}_{z=z_n}\frac{f(z)}{z-z_0}} \in \mathbb{R}$$

$$I = \int\frac{\sin(a\cdot x)}{x}dx = (-i)\,\mathcal{P}\int\frac{e^{+i a\cdot x}}{x}dx = (-i)\,\mathcal{P}\int\frac{\cos(ax)+i\sin(ax)}{x}dx$$

$$= \pi\cdot i\,(-i) = \pi$$

$$\mathcal{P}\int\underbrace{\frac{\cos(ax)}{x}}_{\tilde g(x)}dx = 0$$

$\tilde g(x) \to \tilde g(-x) = -\tilde g(x)$



6.2. Retardierte Potentiale

$$\left.\begin{array}{l}\Box\,\Phi = -4\pi\,\overset{\text{Ladungsdichte}}{\rho} \\ \Box\,\underline{A} = -\frac{4\pi}{c}\,\underset{\text{Stromdichte}}{\underline{j}}\end{array}\right\}\ \text{WG für Pot. in Lorenz-Eichung}$$

einfacher

$$\Box\,\Psi = \left\{\Delta - \frac{1}{c^2}\partial_t^2\right\}\Psi(\underline{r},t) = -4\pi\cdot f(\underline{r},t)$$

$$\boxed{\left\{\Delta - \frac{1}{c^2}\partial_t^2\right\}G(\underline{r},t,\underline{r}',t') = -4\pi\,\delta(\underline{r}-\underline{r}')\,\delta(t-t')}$$

Poisson-Int.: $\Phi(\underline{r}) = \int\frac{\overset{\text{Quelle}}{\rho(\underline{r}')}}{\underset{\text{GF}}{|\underline{r}-\underline{r}'|}}d^3r'$

Wenn G gefunden ist

Translationsinvarianz: $G(\underline{r},t,\underline{r}',t') = G(\underline{r}-\underline{r}',t-t')$

$$\Psi(\underline{r},t) = \int d^3r'\,dt'\,G(\underline{r}-\underline{r}',t-t')\,f(\underline{r}',t')$$

betrachte FT $\circ$ $g(\underline{k},\omega) = \int G(\underline{r},t)\,e^{-i(\underline{k}\cdot\underline{r}-\omega t)}\,d^3r\,dt$

$$G(\underline{r},t) = \frac{1}{(2\pi)^4} \int g(\underline{k},\omega)\, e^{+i(\underline{k}\cdot\underline{r} - \omega t)}\, d^3k\, d\omega$$

$$\circ\ \delta^{(3)}(\underline{r}-\underline{r}')\,\delta(t-t') = \frac{1}{(2\pi)^4} \int e^{+i[\underline{k}(\underline{r}-\underline{r}') - \omega(t-t')]}\, d^3k\, d\omega$$

$$\int g(\underline{k},\omega)\, \Box\, e^{+i\overbrace{[\underline{k}(\underline{r}-\underline{r}') - \omega(t-t')]}^{}_{[\dots]}}\, d^3k\, d\omega = \int g(\underline{k},\omega) \left[\frac{1}{c^2}\cdot\omega^2 - \underline{k}\cdot\underline{k}\right] e^{+i[\dots]}$$

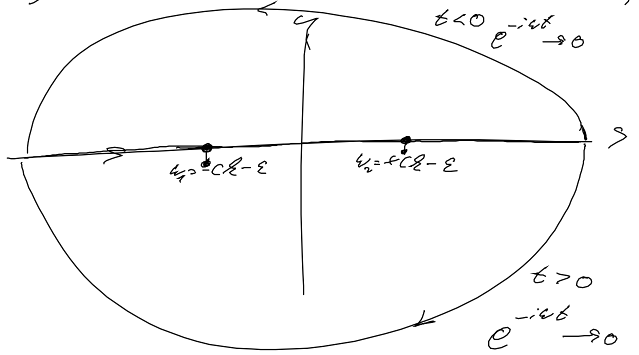
$$= -4\pi \int [1]\, e^{+i[\dots]}$$

$$\Rightarrow g(k,\omega) = \frac{4\pi}{k^2 - \frac{\omega^2}{c^2}}$$

$$\Rightarrow G(\underline{r},t) = \frac{1}{4\pi^3} \int \frac{c^2}{c^2k^2 - \omega^2}\, e^{+i[\underline{k}\cdot\underline{r} - \omega t]}\, d^3k\, d\omega$$

$$= \frac{1}{4\pi^3} \int d^3k\, e^{i\underline{k}\cdot\underline{r}} \cdot \underline{I}(k,t)$$

$$\underline{I}(k,t) = \int \frac{e^{-i\omega t}\, c^2}{c^2k^2 - \omega^2}\, d\omega = -\int \frac{c^2 e^{-i\omega t}}{(\omega - ck)(\omega + ck)}\, d\omega$$



per Hand: Randbedingung

Folge muss nach der Ursache erfolgen

"Kausalität"

$$G(\underline{r}, t<0) \overset{!}{=} 0$$

$$\underline{I}(k,t) = \lim_{\varepsilon\to 0} \int \frac{-c^2\, e^{-i\omega t}}{(\omega - ck + i\varepsilon)(\omega + ck + i\varepsilon)} \quad \to \underline{I}(k, t<0) = 0 \Rightarrow$$

$$= \Theta(t)\cdot 2\pi i \left[\operatorname*{Res}_{\omega = +ck - i\varepsilon} \frac{c^2 e^{-i\omega t}}{(\omega - ck + i\varepsilon)(\omega + ck + i\varepsilon)} + \operatorname*{Res}_{\omega = -ck - i\varepsilon} \frac{c^2 e^{-i\omega t}}{(\omega - ck + i\varepsilon)(\omega + ck + i\varepsilon)}\right]$$

$$= \Theta(t)\cdot 2\pi i \left[\frac{c^2 e^{-ick\cdot t}}{2ck} + \frac{c^2 e^{+ickt}}{-2ck}\right]$$

$$= 2\pi \cdot \Theta(t) \cdot \frac{c}{k} \cdot \sin(c\cdot k\cdot t)$$

$$G(\underline{r},t) = \frac{1}{2\pi^2}\Theta(t) \int_0^\infty k^2\, dk \int_{-1}^{+1} dx \int_0^{2\pi} d\varphi\, \frac{c}{k} \sin(c\cdot k\cdot t)\, e^{+i k\cdot r\cdot x}$$

$$= \frac{2}{\pi}\, \Theta\, \frac{c}{r} \frac{1}{2} \int_{-\infty}^{\infty} \underbrace{\sin(c\cdot k\cdot t)}_{\text{analog}}\, \underbrace{\sin(k\cdot r)}_{\frac{1}{2i}\left(e^{+ikr} - e^{-ikr}\right)}\, dk$$

$$= \frac{-1}{4\pi}\,\Theta(t)\cdot\frac{c}{r}\cdot(2\pi)\cdot(2)\left[\delta(r+ct)-\underbrace{\delta(r-ct)}_{\text{trägt bei}}\right]$$

$$= \frac{c}{r}\,\delta(r-c\cdot t) = \boxed{\frac{1}{r}\cdot\delta\left(\frac{|\underline{r}|}{c}-t\right) = G(\underline{r},t)}$$

$$\boxed{G(r-r',t-t') = \frac{\delta\left(t-t'-\frac{|\underline{r}-\underline{r}'|}{c}\right)}{|r-r'|}}$$

GF d. WG in $(3+1)$d

$$\Psi(r,t) = \int d^3r'\,dt'\,\frac{f(r',t')}{|r-r'|}\,\delta\left(t-t'-\frac{|r-r'|}{c}\right)$$

$$\boxed{\begin{aligned}\underline{\Phi}(r,t) &= \int d^3r'\,\frac{\rho\left(r',t-\frac{|r-r'|}{c}\right)}{|r-r'|}\\ \underline{A}(\underline{r},t) &= \frac{1}{c}\int d^3r'\,\frac{\underline{j}\left(r',t-\frac{|r-r'|}{c}\right)}{|r-r'|}\end{aligned}}$$

„retardierte Potentiale"

$$t' = t - \frac{|r-r'|}{c}$$