

Wdh + Leiter-Oberfl. sind Äquipot.-Fl.

$E$  steht  $\perp$  auf Leiter-Oberfl.

+ Greensche Fkt Lösung der Poisson-Gl. für eine Punktladung

+ Bsp. Dirichlet RB  $\rightarrow$  wähle  $G_D(r, r') = 0$  für alle  $r' \in V$

$$\Rightarrow \Phi(r \in V) = \int_V \rho(r') \underbrace{G_D(r, r')}_{\substack{\text{vorgegeben} \\ \text{mit Randwert}}} d^3r - \frac{1}{4\pi\epsilon_0} \oint_{\partial V} \underbrace{\Phi(r')}_{\substack{\text{mit Randwert} \\ \text{Dirichlet-RB}}} \underbrace{\frac{\partial G_D}{\partial n'}}_{\text{Dirichlet-RB}} dS'$$

Konstante:  $\frac{\partial \Phi}{\partial n} = (\nabla \Phi) \cdot \underline{n}$

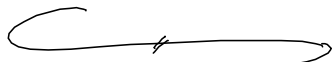
+ Methode der Bildladungen: RB werden durch imag. LV ersetzt

+ Bsp. Ebene

$$G_D(r, r') = \frac{1}{|r - r'|} - \frac{1}{|r - r'_B|}$$

$\uparrow$   
Ort der PL

$$r' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \rightarrow r'_B = \begin{pmatrix} x' \\ y' \\ -z' \end{pmatrix} \notin V$$

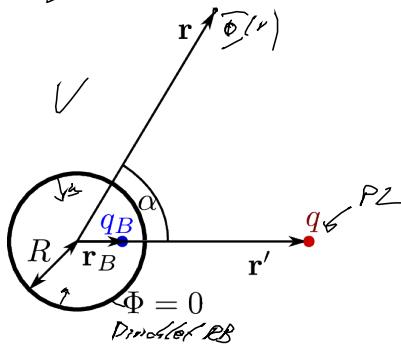


1. 11.2. Beispiel: PL vor (geerdeter) Kugel

Volumen:  $V = \{r: |r| \geq R\}$

RB:  $\Phi(|r|=R) = 0$

$\Phi(|r| \rightarrow \infty) = 0$



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Ausatz:  $\Phi(r) = \frac{q}{|r-r'|} + \frac{q_B}{|r-r_B'|}$   $\underline{e}_r \cdot \underline{e}_{r'} = \cos \alpha$

$$= \frac{q}{r} + \frac{q_B/r_B'}{|r/r_B' - \underline{e}_r \cdot \underline{e}_{r'}|}$$

$\Phi(r=R) = \frac{q/R}{\sqrt{1 + (\frac{r'}{R})^2 - 2 \frac{r'}{R} \cos \alpha}} + \frac{q_B/r_B'}{\sqrt{(\frac{R}{r_B'})^2 + 1 - 2 \frac{R}{r_B'} \cos \alpha}} = 0$

$$\frac{q}{R} = - \frac{q_B}{r_B'} \quad \frac{r'}{R} = \frac{R}{r_B'} \quad r' > R$$

$$q_B = - \frac{R}{r'} \cdot q \quad r_B' = \frac{R}{r'} \cdot R \rightarrow r_B' < R$$

$$|q_B| < |q|$$

$$\Phi(r) = q \left( \frac{1}{|r-r'|} - \frac{R}{r'} \frac{1}{|r - \frac{R^2}{r'} \underline{r}'|} \right)$$

berechne  $E = -\nabla \Phi$  an der Kugel-Oberfl.

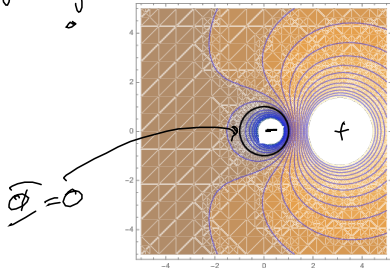
$$E = -\frac{1}{\epsilon_0} \cdot (\nabla \Phi) \Big|_{r=R} = -\frac{1}{\epsilon_0} \frac{\partial \Phi}{\partial r} \Big|_{r=R}$$

$$= -\frac{1}{\epsilon_0} \frac{q}{R^2} \cdot \frac{R}{r'} \frac{1}{\left[1 + (\frac{R}{r'})^2 - 2 \frac{R}{r'} \cos \alpha\right]^{3/2}}$$

Flächen-Ladungsdichte

+ rot.-symm. um  $\alpha$   
+ normal für  $\alpha=0$

$$+ \int_0^\pi d\varphi \int_0^\pi d\alpha R^2 \sin \alpha E = -q \frac{R}{r'} = q_B$$



$$G_D(r, r') = \frac{1}{|r-r'|} - \frac{1}{|\frac{r'}{R} r - \frac{R}{r'} r'|}$$

$$= \frac{1}{\sqrt{r^2 + r'^2 - 2rr' \underline{e}_r \cdot \underline{e}_{r'}}} - \frac{1}{\sqrt{\frac{(r')^2 r^2}{R^2} + R^2 - 2r r' \underline{e}_r \cdot \underline{e}_{r'}}$$

$$G_D(r, r') = G_D(r', r) \quad G_D(r, r') = 0 \quad \forall r, r' \in \partial V$$

$$\left. \frac{\partial G_D}{\partial r'} \right|_{\partial V} = - \left. \frac{\partial G_D}{\partial r'} \right|_{r'=R} = -\frac{1}{R} \frac{r^2 - R^2}{[r^2 + R^2 - 2rR \cos(\vartheta - \vartheta')]^{3/2}}$$

$$\begin{aligned} \Phi(r) &= \int_V \rho(r') G_D(r, r') d^3r' \\ &\quad - \frac{1}{4\pi} \oint_{\partial V} \Phi(r') \frac{\partial G}{\partial n'} dS' \\ &= \dots \end{aligned}$$

## 2. Multipolentwicklung

### 2.1. Vollst. Funktionensysteme

$$f(x) = \sum_n \underbrace{a_n}_{\text{Entwickl. Koeffizienten}} \underbrace{g_n(x)}_{\text{Basis-Fkt}}$$

Eine Basis  $\{\tilde{g}_n(x)\}$  heißt auf dem Intervall  $I = [a, b]$  orthogonal

$$\int_a^b \tilde{g}_n^*(x) \tilde{g}_m(x) dx = \mu_n \cdot \delta_{nm} \quad \mu_n \neq 0$$

Normierung  $g_n(x) = \frac{1}{\sqrt{\mu_n}} \tilde{g}_n(x)$

$$\boxed{\int_a^b g_n^*(x) g_m(x) dx = \delta_{nm} \quad \text{ortho normales Fkt.-system}}$$

Norm einer Fkt  $\|f(x)\|^2 = \int_a^b |f(x)|^2 dx$

$$f(x) = \sum_n a_n \cdot g_n(x)$$

$$a_n = \int_a^b g_n^*(x) f(x) dx$$

Ein ODE-FS heißt vollständig, falls sich jede quadrat-integrierte Fkt in dieser Basis darstellen lässt

$$f(x) = \sum_n \left[ \int_a^b g_n^*(y) f(y) dy \right] \cdot g_n(x)$$

$$= \int_a^b dy \left[ \sum_n g_n^*(y) \cdot f(y) \cdot g_n(x) \right]$$

$$\sum_n g_n^*(y) \cdot g_n(x) = \delta(x-y) \quad \forall x, y \in [a, b]$$

Vollständigkeitsrelation

### 2.1.1 Fourier-Reihen

$$\begin{aligned} g_n(x) &= \frac{1}{\sqrt{L}} \sin(n \cdot x) \\ h_n(x) &= \frac{1}{\sqrt{L}} \cdot \cos(n \cdot x) \\ h_0(x) &= \frac{1}{\sqrt{2L}} \end{aligned} \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} n \in \{1, 2, \dots\}$$

kur ungerade Fkt

kur gerade Fkt

$I = [-\bar{L}, +\bar{L}]$  z.B.  $\int_{-\bar{L}}^{+\bar{L}} \frac{1}{L} \sin(n \cdot x) \cos(n \cdot x) dx = 0$

$$\int_{-\bar{L}}^{+\bar{L}} |h_0(x)|^2 dx = 1$$

jede q-i Fkt lässt sich in  $[-\bar{L}, +\bar{L}]$  darstellen als

$$f(x) = \sum_{n=0}^{\infty} a_n \cdot h_n(x) + \sum_{n=1}^{\infty} b_n \cdot g_n(x)$$

$$a_0 = \frac{1}{\sqrt{2L}} \int_{-\bar{L}}^{+\bar{L}} f(x) dx$$

$$a_{n \geq 1} = \frac{1}{\sqrt{L}} \int_{-\bar{L}}^{+\bar{L}} f(x) \cdot \cos(n \cdot x) dx$$

$$b_{n \geq 1} = \frac{1}{\sqrt{L}} \int_{-\bar{L}}^{+\bar{L}} f(x) \sin(n \cdot x) dx$$

Alternativ andere Basis

$$E_n(x) = \frac{1}{\sqrt{2L}} e^{i n \cdot x}$$

$$f(x) = \sum_n C_n E_n(x)$$

$$n \in \mathbb{Z}$$

$$C_n = \frac{1}{\sqrt{2L}} \int_{-\bar{L}}^{+\bar{L}} e^{-i n \cdot x} \cdot f(x) dx$$

## 2.1.2. Legendre-Polynome

$$\gamma = \frac{r}{|r-r'|} = \frac{r}{\sqrt{r^2+r'^2-2rr'\cos\alpha}} = \frac{r}{\sqrt{r_2^2+r_1^2-2r_2r_1\cos\alpha}}$$

$$\alpha = \angle(r, r')$$

$$r_2 = \min(r, r')$$

$$r_1 = \max(r, r')$$

$$\frac{r_2}{r_1} \leq 1$$

Bsp  $\Phi(r) = \int \frac{\rho(r')}{|r-r'|} d^3r'$   
 $r \gg r'$

$$\gamma = \frac{1}{r_1 \sqrt{1 - 2 \frac{r_2}{r_1} \cos\alpha + \left(\frac{r_2}{r_1}\right)^2}}$$

$$= \frac{1}{r_1} \left[ 1 + \cos\alpha \frac{r_2}{r_1} + \frac{1}{2} (3\cos^2\alpha - 1) \left(\frac{r_2}{r_1}\right)^2 + \dots \right]$$

$$= \frac{1}{r_1} \sum_{n=0}^{\infty} P_n(\cos\alpha) \left(\frac{r_2}{r_1}\right)^n$$

### Legendre-Polynome

die ersten LP's sind

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

gerade  $\alpha x$

$$x \in [-1, +1]$$

ungerade  $\alpha x$

Rodriguez-Formel

$$P_n(x) = \frac{1}{2^n \cdot n!} \frac{d^n}{dx^n} (x^2-1)^n$$

Wdh. ausprobieren

$$g(t, x) = \frac{1}{\sqrt{1-2xt+t^2}}$$

$$= \sum_{n=0}^{\infty} P_n(x) \cdot t^n$$

"generierende Fkt" der LP

$$\frac{\partial g}{\partial t} = \frac{x-t}{[1-2xt+t^2]^{3/2}} = \sum_n n \cdot P_n(x) \cdot t^{n-1}$$

$$(1-2xt+t^2) \sum_n n \cdot P_n(x) t^{n-1} = (x-t) \sum_n P_n(x) t^n$$

$$\sum_{n=0}^{\infty} n \cdot P_n(x) t^{n-1} - \sum_{n=0}^{\infty} 2n \cdot x \cdot P_n(x) t^n + \sum_{n=0}^{\infty} n \cdot P_n(x) t^{n+1}$$

$$= \sum_{s=0}^{\infty} P_s(x) t^{s+1} - \sum_{n=0}^{\infty} x \cdot P_n(x) t^n = 0$$

$$n = n+1 \quad s = n-1$$

$$0 = (k+1) \cdot P_{k+1}(x) - (2k+1)xP_k(x) + k \cdot P_{k-1}(x) \quad \forall k = 1, 2, \dots$$

⇒ rek. stabile Rekursionsformel

$$P_{k+1}(x) = \frac{2k+1}{k+1} x \cdot P_k(x) - \frac{k}{k+1} P_{k-1}(x)$$

$$= 2x \cdot P_k(x) - P_{k-1}(x) - \frac{x P_k(x) - P_{k-1}(x)}{k+1}$$

generelle Relation zwischen  $P_k(x)$ ,  $P_k'(x)$ ,  $P_k''(x)$

$$\frac{\partial}{\partial x} \frac{t}{\sqrt{1-2xt+t^2}} = \sum_{k=0}^{\infty} P_k'(x) \cdot t^k$$

weder das selbe  $\cdot P_{k+1}(x) + P_{k-1}(x) = P_k(x) + 2x P_k'(x)$

$\cdot$  Rek. Formel ableiten

$$\frac{d}{dx} [(2k+1)x \cdot P_k(x)] = (k+1)P_{k+1}(x) + k \cdot P_{k-1}(x)$$

$$\Rightarrow P_{k+1}(x) = (k+1) \cdot P_k(x) + x \cdot P_k'(x) \quad (k \rightarrow k-1)$$

$$P_{k-1}(x) = -k P_k(x) + x \cdot P_k'(x)$$

$$\Rightarrow \frac{(1-x^2)P_k''(x) - 2xP_k'(x)}{(1-x^2)P_k'' - 2xP_k' = k \cdot P_{k+1} - k P_k - k \cdot x \cdot P_k'}$$

$$\frac{(1-x^2)P_k''(x) - 2xP_k'(x)}{\text{Legendre-DGL}} + \frac{k(k+1)P_k(x)}{\text{Legendre-DGL}} = 0; k \in \{0, 1, 2, \dots\}$$

zeige: Orthogonalität

DGL schreiben als  $\frac{d}{dx} [(1-x^2)P_k'(x)] = -k(k+1)P_k(x)$

$$\int_{-1}^1 P_k(x) \frac{d}{dx} [(1-x^2)P_k'(x)] dx = -k(k+1) \int_{-1}^1 P_k(x) P_k(x) dx$$

$k \leftrightarrow k$  und ableiten

$$\int_{-1}^1 \left\{ P_k(x) \frac{d}{dx} [(1-x^2)P_k'(x)] - P_k(x) \frac{d}{dx} [(1-x^2)P_k'(x)] \right\} dx$$

$$= [k(k+1) - k(k+1)] \int_{-1}^1 P_k(x) P_k(x) dx$$

$$0 = \int_{-1}^1 P_k(x) P_k(x) dx$$

$k \neq k: \neq 0$

für  $k \neq k$  gilt:  $\int_{-1}^1 P_k(x) P_k(x) dx = 0$

LP sind orthogonal