Wdh + Leiter-Oberfl. sind Äquipot.-Fl.
E steht ⊥ auf Leiter-Oberf.

+ Greensche Fkt: Lösung der Poisson-Gl. für eine Punktladung

+ Bsp. Dirichlet RB → wähle $G_D(r,r') = 0$ für alle $r' \in V$

$$\Rightarrow \underset{\sim}{\Phi}(r \in V) = \int_V \rho(r') \underline{G_D(r,r')} d^3r' - \frac{1}{4\pi} \oint_{\partial V} \Phi(r') \underline{\frac{\partial G_D}{\partial n'}}(r,r') dS'$$

vorgegeben — gut zu haben — Dirichlet-RB

Normalen-Abl. $\frac{\partial \psi}{\partial n} = (\nabla \psi) \cdot \underline{n}$

+ Methode der Bildladungen: RB werden durch sog. LV ersetzt

+ Bsp. Ebene

$$G_D(r,r') = \frac{1}{|r-r'|} - \frac{1}{|r - r_B'|}$$

Ort der PL

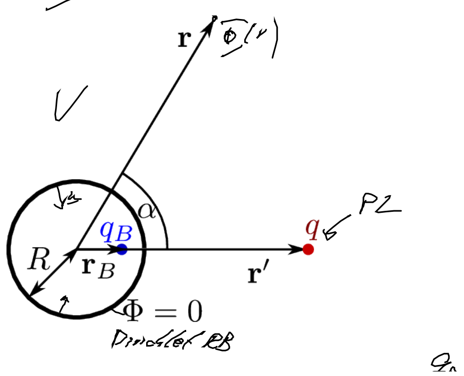
$$r' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \longrightarrow r_B' = \begin{pmatrix} x' \\ y' \\ -z' \end{pmatrix} \notin V$$

1.14.2. Beispiel: PL vor (geerdeter) Kugel

Volumen: $V = \{\underline{r} : |r| \geq R\}$

RB: $\underset{\sim}{\Phi}(|r| = R) = 0$

$\underset{\sim}{\Phi}(|r| \to \infty) = 0$



Ansatz: $\Phi(r) = \frac{q}{|\underline{r}-\underline{r}'|} + \frac{q_B}{|\underline{r}-\underline{r}_B'|}$ $\qquad \underline{e}_r \cdot \underline{e}_{r'} = \cos\alpha$

$$= \frac{q/r}{|\underline{e}_r - \frac{r'}{r}\underline{e}_{r'}|} + \frac{q_B/r_B'}{|\frac{r}{r_B'}\underline{e}_r - \underline{e}_{r'}|}$$

$$\Phi(r=R) = \frac{q/R}{\sqrt{1+\left(\frac{r'}{R}\right)^2 - 2\frac{r'}{R}\cos\alpha}} + \frac{q_B/r_B'}{\sqrt{\left(\frac{R}{r_B'}\right)^2 + 1 - 2\frac{R}{r_B'}\cdot\cos\alpha}} = 0$$

$$\frac{q}{R} = -\frac{q_B}{r_B'} \qquad \frac{r'}{R} = \frac{R}{r_B'} \qquad r' > R$$

$$q_B = -\frac{R}{r'}\cdot q \qquad r_B' = \frac{R}{r'}\cdot R \longrightarrow r_B' < R$$

$$|q_B| < |q|$$

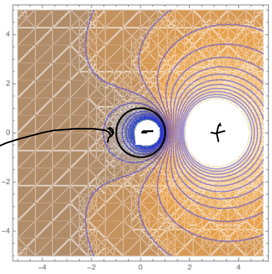
$$\Phi(r) = q\left(\frac{1}{|\underline{r}-\underline{r}'|} - \frac{\frac{R}{r'}}{|\underline{r} - \frac{R^2}{(r')^2}\cdot\underline{r}'|}\right)$$

berechne $E = -\nabla\Phi$ an der Kugel-Oberfl.

$$\sigma = -\frac{1}{4\pi}\underline{n}\cdot(\nabla\Phi)\Big|_{r=R} = -\frac{1}{4\pi}\frac{\partial\Phi}{\partial r}\Big|_{r=R}$$

$$= -\frac{1}{4\pi}\frac{q}{R^2}\cdot\frac{R}{r'}\frac{\left(1-\left(\frac{R}{r'}\right)^2\right)}{\left[1+\left(\frac{R}{r'}\right)^2 - 2\left(\frac{R}{r'}\right)\cos\alpha\right]^{3/2}}$$ Influenz-Ladungsdichte

+ rot.-symm. um α
+ maximal für $\alpha = 0$
+ $\int_0^{2\pi} d\varphi \int_0^{\pi} d\alpha\, R^2 \sin\alpha\, \sigma = -q\frac{R}{r'} = q_B$



$\Phi = 0$

$$G_D(r, r') = \frac{1}{|\underline{r}-\underline{r}'|} - \frac{1}{|\frac{r'}{R}\underline{r} - \frac{R}{r'}\underline{r}'|}$$

$$= \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\underline{e}_r\cdot\underline{e}_{r'}}} - \frac{1}{\sqrt{\frac{(r')^2 r^2}{R^2} + R^2 - 2rr'\underline{e}_r\cdot\underline{e}_{r'}}}$$

$$G_D(r, r') = G_D(r', r) \qquad G_D(r, r') = 0 \quad \forall r, r' \in \partial V$$

$$\left.\frac{\partial G_D}{\partial n'}\right|_{\partial V} = -\left.\frac{\partial G_D}{\partial r'}\right|_{r'=R} = -\frac{1}{R}\frac{r^2-R^2}{[r^2+R^2-2rR\,\underline{e_r}\cdot\underline{e_{r'}}]^{3/2}}$$

$$\Phi(r) = \int \varrho(r')\, G_D(r,r')\, d^3r' - \frac{1}{4\pi}\oiint_{\partial V} \Phi(r')\frac{\partial G}{\partial n'}\, dS'$$

$$= \dots$$

2. Multipolentwicklungen

2.1. Vollst. Funktionensysteme

$$f(x) = \sum_n a_n\, g_n(x)$$

Entw.-koeffizienten   Basis-Fkt

Eine Basis $\{\tilde g_n(x)\}$ heißt auf dem Intervall $I = [a,b]$ orthogonal

$$\int_a^b \tilde g_n^*(x)\, \tilde g_m(x)\, dx = N_n \cdot \delta_{nm}$$

$$N_n \neq 0$$

Normierung $\quad g_n(x) = \frac{1}{\sqrt{N_n}}\,\tilde g_n(x)$

$$\boxed{\int_a^b g_n^*(x)\, g_m(x)\, dx = \delta_{nm}}$$

orthonormales Fkt.-system

Norm einer Fkt $\quad \|f(x)\|^2 = \int_a^b |f(x)|^2\, dx$

$$f(x) = \sum_n a_n \cdot g_n(x)$$

$$a_n = \int_a g_n^*(x) f(x)\, dx$$

Ein ON-FS heißt vollständig, falls sich jede quadrat-integrable Fkt in diese Basis darstellen lässt

$$f(x) = \sum_n \left[\int_a^b g_n^*(q) f(q)\, dq \right] \cdot g_n(x)$$

$$= \int_a^b dq \left[\sum_n g_n^*(q) \cdot f(q) \cdot g_n(x) \right]$$

$$\boxed{\sum_n g_n^*(q) \cdot g_n(x) = \delta(x-q) \qquad \forall\, x,q \in [a,b]}$$

Vollständigkeitsrelation

2.1.1 Fourier-Reihen

$$g_n(x) = \frac{1}{\sqrt{\pi}} \sin(n \cdot x)$$

$$h_n(x) = \frac{1}{\sqrt{\pi}} \cdot \cos(n \cdot x)$$

$$n \in \{1, 2, \dots\}$$

$$h_0(x) = \frac{1}{\sqrt{2\pi}}$$

$$I = [-\pi, +\pi] \qquad \text{z.B.} \int_{-\pi}^{+\pi} \frac{1}{\pi} \sin(nx) \cos(nx) = 0$$

$$\int_{-\pi}^{+\pi} |h_0(x)|^2 dx = 1$$

nur ungerade Fkt

nur gerade Fkt

jede q-i Fkt lässt sich in $[-\pi, +\pi]$ darstellen als

$$f(x) = \sum_{n=0}^{\infty} a_n \cdot h_n(x) + \sum_{n=1}^{\infty} b_n \cdot g_n(x)$$

$$a_0 = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{+\pi} f(x)\, dx$$

$$a_{n \geq 1} = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{+\pi} f(x) \cdot \cos(nx)\, dx$$

$$b_{n \geq 1} = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{+\pi} f(x) \sin(nx)\, dx$$

Alternativ andere Basis

$$e_n(x) = \frac{1}{\sqrt{2\pi}} e^{i n \cdot x} \qquad n \in \mathbb{Z}$$

$$f(x) = \sum_n c_n\, e_n(x) \qquad c_n = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{+\pi} e^{-inx} \cdot f(x)\, dx$$

2.1.2. Legendre-Polynome

$$\frac{1}{|\underline{r}-\underline{r}'|} = \frac{1}{\sqrt{r^2+r'^2-2rr'\cos\alpha}} = \frac{1}{\sqrt{r_<^2+r_>^2-2r_<\cdot r_>\cdot\cos\alpha}}$$

$\alpha = \sphericalangle(\underline{r},\underline{r}')$

$r_< = \min(r,r')$ $\qquad \frac{r_<}{r_>} \leq 1$ $\qquad$ Bsp $\phi(r) = \int \frac{\rho(r')}{|\underline{r}-\underline{r}'|} d^3r'$

$r_> = \max(r,r')$ $\qquad\qquad\qquad\qquad r \gg r'$

$$\frac{1}{\ldots} = \frac{1}{r_>\sqrt{1-2\underbrace{\frac{r_<}{r_>}}_{t}\cos\alpha+\left(\frac{r_<}{r_>}\right)^2}}$$

$$= \frac{1}{r_>}\left[1+\cos\alpha\frac{r_<}{r_>}+\frac{1}{2}(3\cos^2\alpha-1)\left(\frac{r_<}{r_>}\right)^2+\ldots\right]$$

$$= \frac{1}{r_>}\sum_{n=0}^{\infty}\underbrace{P_n(\cos\alpha)}_{\text{Legendre-Polynome}}\left(\frac{r_<}{r_>}\right)^n$$

die ersten LP's sind

$$\left.\begin{aligned}P_0(x) &= 1\\ P_1(x) &= x\\ P_2(x) &= \frac{1}{2}(3x^2-1)\\ P_3(x) &= \frac{1}{2}(5x^3-3x)\end{aligned}\right.$$

gerade in x (P_0, P_2)

ungerade in x (P_1, P_3)

$x \in [-1,+1]$

Rodriguez-Formel

$$\boxed{P_n(x) = \frac{1}{2^n\cdot n!}\frac{d^n}{dx^n}(x^2-1)^n}$$

kau. unpraktisch

$$g(t,x) = \frac{1}{\sqrt{1-2x\cdot t+t^2}}$$ „Generierende Fkt" der LP

$$= \sum_{n=0}^{\infty} P_n(x)\cdot t^n$$

$$\frac{\partial g}{\partial t} = \frac{x-t}{[1-2xt+t^2]^{3/2}} = \sum_n n\cdot P_n(x)\cdot t^{n-1}$$

$$(1-2xt+t^2)\sum_n n\cdot P_n(x)\,t^{n-1} = (x-t)\sum_n P_n(x)\,t^n$$

$$\sum_{n=0}^{\infty} n\cdot P_n(x)t^{n-1} - \sum_{n=0}^{\infty}2n\cdot x\,P_n(x)\,t^n + \sum_{s=0}^{\infty}s\cdot P_s(x)t^{s+1}$$
$$+\sum_{s=0}^{\infty}P_s(x)\,t^{s+1} - \sum_{n=0}^{\infty}x\cdot P_n(x) \qquad = 0$$

$n = n+1 \qquad s = k-1$

$$0 = (n+1)\cdot P_{n+1}(x) - (2n+1)x P_n(x) + n\cdot P_{n-1}(x) \qquad \forall n = 1, 2, \ldots$$

$\Rightarrow$ num. stabile Rekursionsformel

$$P_{n+1}(x) = \frac{2n+1}{n+1}\, x\cdot P_n(x) - \frac{n}{n+1} P_{n-1}(x)$$
$$= 2x\cdot P_n(x) - P_{n-1}(x) - \frac{x P_n(x) - P_{n-1}(x)}{n+1}$$

brauchen Relation zwischen $P_n(x)$, $P_n'(x)$, $P_n''(x)$

$$\frac{\partial g}{\partial x} = \frac{t}{[1-2xt+t^2]^{3/2}} = \sum_{n=0}^{\infty} P_n'(x)\cdot t^n$$

wieder dasselbe · $P_{n+1}'(x) + P_{n-1}'(x) = P_n(x) + 2x P_n'(x)$

· Rek.-Formel ableiten

$$\frac{d}{dx}\left[(2n+1)\cdot x\cdot P_n(x)\right] = (n+1)P_{n+1}'(x) + n\cdot P_{n-1}'(x)$$

$$\Rightarrow P_{n+1}'(x) = (n+1)\cdot P_n(x) + x\cdot P_n'(x) \qquad (n \to n-1)$$
$$\underbrace{P_{n-1}'(x) = -n P_n(x) + x\cdot P_n'(x)}$$

$$\Rightarrow (1-x^2)P_n'(x) = n\cdot P_{n-1}(x) - n\cdot x P_n(x)$$
$$(1-x^2)P_n'' - 2x P_n' = n\cdot P_{n-1}' - n P_n - n\cdot x\cdot P_n'$$

$$\boxed{(1-x^2)P_n''(x) - 2x P_n'(x) + n(n+1)P_n(x) = 0; n \in \{0, 1, 2, \ldots\} \quad \text{Legendre-DGL}}$$

zeige: Orthogonalität

DGL schreiben als $\frac{d}{dx}\left[[1-x^2]\cdot P_n'(x)\right] = -n(n+1)\cdot P_n(x)$

$$\int_{-1}^{+1} P_m(x)\frac{d}{dx}\left[(1-x^2)P_n'(x)\right]dx = -n(n+1)\int P_m(x)P_n(x)dx$$

$n \leftrightarrow m$ und abziehen

$$\int_{-1}^{+1}\left\{P_m(x)\frac{d}{dx}\left[(1-x^2)P_n'(x)\right] - P_n(x)\frac{d}{dx}\left[(1-x^2)P_m'(x)\right]\right\}dx$$
$$= \left[m(m+1) - n(n+1)\right]\cdot\int_{-1}^{+1} P_m(x)P_n(x)\,dx$$

$$0 = \underbrace{[\quad]}_{n\neq m\,:\,\neq 0}\int_{-1}^{+1} P_m(x)P_n(x)\,dx$$

für $n \neq m$ gilt: $\int P_m(x)P_n(x)dx = 0$

LP sind orthogonal