

English Summary:

classical Van der Pol osc.: $\ddot{Q} + \omega_0^2 Q - \epsilon(1 - 2Q^2)\dot{Q} = 0 \quad \epsilon > 0$

neg. lin. damping pos. nonlin. damping

network of VdP osc. (nonlocal coupling of range R):

$$\dot{x}_k = F(x_k) + \frac{\epsilon}{2R} \sum_{j=k-R}^{k+R} H(x_j - x_k), \quad k=1, \dots, N, \quad x_k = \begin{pmatrix} Q \\ P \end{pmatrix} \in \mathbb{R}^2, \quad P = \dot{Q}$$

H 2x2 coupling matrix

⇒ chimera states

$\epsilon \ll 1$: $\dot{\alpha} = \frac{\epsilon}{2}(1 - |\alpha|^2)\alpha$ Stuart-Landau osc. $\alpha \equiv \frac{1}{\sqrt{2}}(Q + iP)$

Quantum Van der Pol osc.:

$$\dot{\rho}_S = 2\kappa_1 (a^\dagger \rho_S a - \frac{1}{2} \{ \rho_S, a a^\dagger \}) + 2\kappa_2 (a^2 \rho_S (a^\dagger)^2 - \frac{1}{2} \{ \rho_S, (a^\dagger)^2 a^2 \})$$

1-photon absorption 2-photon emission

4.7.2 Quantum Van der Pol oscillator

Bastidas, Omelchenko, Zakharov, Schöll, Brandes, PRE 92, 062924 (28.12.2015)

-||- , in "Control of Self-Organizing Nonlinear Systems" (eds. Schöll, Klapp, Hövel) Springer 2016

Aim: Lindblad Master eq. with dissipation rates κ_1 (linear neg. damping) and κ_2 (nonlinear loss)

Decomposition of bosonic operators

$$a(t) = \tilde{a} + \alpha(t) \quad a^\dagger(t) = \tilde{a}^\dagger + \alpha^*(t)$$

quantum fluctuations
annih. op. \tilde{a}
mean field
 $\alpha = \langle \alpha | a | \alpha \rangle \in \mathbb{C}$
semiclass. trajectory
coherent state α
 $a | \alpha \rangle = \alpha | \alpha \rangle$

Quantum fluctuations about a semiclassical trajectory $\alpha(t)$:

Density matrix in the co-moving frame (co-rotating with $\alpha(t)$):

$$\rho_\alpha(t) = D^\dagger(\alpha) \rho(t) D(\alpha) \quad \rho \equiv \rho_S$$

with displacement op. $D(\alpha) = e^{\alpha a^\dagger - \alpha^* a} = e^{-\frac{|\alpha|^2}{2}} e^{\alpha \tilde{a}^\dagger - \alpha^* \tilde{a}} e^{\frac{|\alpha|^2}{2}}$
 (unitary)

$$\dot{\rho}_\alpha(t) = -\frac{i}{\hbar} [H^{(K)}(t), \rho_\alpha(t)] + 2\kappa_1 D^\dagger(\alpha) (\tilde{a}^\dagger \rho \tilde{a} - \frac{1}{2} \{ \rho, \tilde{a} \tilde{a}^\dagger \}) D(\alpha) + 2\kappa_2 D^\dagger(\alpha) (\tilde{a}^2 \rho (\tilde{a}^\dagger)^2 - \frac{1}{2} \{ \rho, (\tilde{a}^\dagger)^2 \tilde{a}^2 \}) D(\alpha)$$

$$= -\frac{i}{\hbar} [H^{(0)}(t), \rho_\alpha(t)] + \hat{\mathcal{L}}_1 \rho_\alpha + \hat{\mathcal{L}}_2 \rho_\alpha$$

Liouville operators of dissipation

with $H^{(0)}(t) := -i\hbar D^\dagger(\alpha) \partial_t D(\alpha)$

$$= -\frac{i\hbar}{2} [\dot{\alpha}(t) \alpha^*(t) - \alpha(t) \dot{\alpha}^*(t)] - i\hbar [\dot{\alpha}(t) \tilde{a}^\dagger - \dot{\alpha}^*(t) \tilde{a}]$$

Gaussian quantum fluctuations for $|\alpha(t)| \gg 1$: semiclassical high-density limit
 \Rightarrow neglect all non-Gaussian (non-quadratic) fluctuations

$$\hat{\mathcal{L}}_1 \rho_\alpha = 2\kappa_1 (\tilde{a}^\dagger \rho_\alpha \tilde{a} - \frac{1}{2} \{ \rho_\alpha, \tilde{a} \tilde{a}^\dagger \}) - \frac{i}{\hbar} [i\hbar \kappa_1 \alpha \tilde{a}^\dagger, \rho_\alpha] - \frac{i}{\hbar} [-i\hbar \kappa_1 \alpha^* \tilde{a}, \rho_\alpha]$$

incoherent (dissip.) part coherent part

lin. in \tilde{a}^\dagger or \tilde{a}

$\hat{\mathcal{L}}_2 \rho_\alpha$ analogously

$$\Rightarrow \dot{\rho}_\alpha(t) = -i\kappa_2 [i(\alpha^*)^2 \tilde{a}^2 - i\alpha^2 (\tilde{a}^\dagger)^2, \rho_\alpha] + 2\kappa_1 (\tilde{a}^\dagger \rho_\alpha \tilde{a} - \frac{1}{2} \{ \rho_\alpha, \tilde{a} \tilde{a}^\dagger \}) + 8\kappa_2 |\alpha|^2 (\tilde{a} \rho_\alpha \tilde{a}^\dagger - \frac{1}{2} \{ \rho_\alpha, \tilde{a}^\dagger \tilde{a} \})$$

quantum fluctuations

position of co-moving frame:

$$\dot{\alpha}(t) = \kappa_1 \alpha(t) - 2\kappa_2 \alpha(t) |\alpha(t)|^2$$

semiclassical trajectory

Interpretation:

initially coherent state $\rho(t=0) = |\alpha(0)\rangle \langle \alpha(0)|$

$\Leftrightarrow \rho_\alpha(t=0) = |0\rangle \langle 0|$ vacuum state in co-moving frame centered at $\alpha(0)$

\uparrow
 $\hat{=}$ initial condition for Eq.(1)

sol. of semiclassical Eq.(2) $\hat{=}$ classical Stuart-Landau eq.

$$\dot{\alpha} = \frac{\epsilon}{2} (1 - |\alpha|^2) \alpha$$

with $\kappa_1 = \frac{\epsilon}{2}$, $\kappa_2 = \frac{\epsilon}{4}$

\Rightarrow (i) limit cycle $|\alpha|^2 = \frac{\kappa_1}{2\kappa_2}$

(ii) time-dependent squeezing described by master eq. (1)

4.7.3 Network of quantum Van der Pol osc.

Ring network of N Van der Pol osc. a_l, a_l^+ , $l=1, \dots, N$
 indices mod N

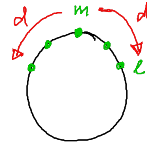
$$(3) \quad \dot{q} = -\frac{i}{\hbar} [H, q] + 2 \sum_{l=1}^N [\kappa_1 \mathcal{D}(a_l^+) + \kappa_2 \mathcal{D}(a_l^2)]$$

with dissipative processes $\mathcal{D}(\hat{O}) := \hat{O} \hat{g} \hat{O}^+ - \frac{1}{2} \{ \hat{g}, \hat{O}^+ \hat{O} \}$ "dissipator"

nonlocal coupling of range d :

new!

$$H = \hbar \sum_{\substack{m=1 \\ m \neq l}}^N K_{lm} (a_l^+ a_m + a_l a_m^+)$$



coupling matrix $K_{lm} = \frac{V}{2d} \Theta(d - |l - m|)$

coupling strength V



general form of Lindblad master eq.

$$\dot{\rho} = -\frac{i}{\hbar} [H, \rho] + \sum_{\mu} \gamma_{\mu} (L_{\mu} \rho L_{\mu}^+ - \frac{1}{2} \{ \rho, L_{\mu}^+ L_{\mu} \})$$

Lindblad op. $L_{\mu} = a_{\mu}^+$ (lin.)
 $L_{\mu} = a_{\mu}^2$ (nonlin.)

eff. Hamiltonian

$$H_{\text{eff}} := H - \frac{i\hbar}{2} \sum_{\mu} \gamma_{\mu} L_{\mu}^+ L_{\mu}$$

$$(3) \Rightarrow H_{\text{eff}} = i\hbar \kappa_1 \sum_{l=1}^N (a_l^+ a_l + 1) - i\hbar \kappa_2 \sum_{l=1}^N \hat{n}_l (\hat{n}_l - 1) \quad \hat{n}_l := a_l^+ a_l$$

$$+ \hbar \sum_{\substack{m=1 \\ m \neq l}}^N K_{lm} (a_l^+ a_m + a_l a_m^+)$$

(nonlocal coupling)

Bose-Hubbard model with long-range interactions
 and complex on-site self-energies and chem. pot.

(\Rightarrow application to driven dissipative Bose-Einstein condensation)

Gaussian quantum fluctuations

expansion $b_l(t) := \mathcal{D}^+(\alpha(t)) a_l \mathcal{D}(\alpha(t)) = \tilde{a}_l + \alpha_l(t)$
quantum fluct. *mean-field*

with displacement op. $\mathcal{D}(\alpha) = e^{\alpha \tilde{a}^+ - \alpha^* \tilde{a}}$

$$\alpha(t) \equiv (\alpha_1, \alpha_2, \dots, \alpha_N)$$

$$\tilde{\alpha} \equiv (\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_N)$$

Semiclassical regime $|\alpha_\ell(t)| \gg \|\tilde{\alpha}_\ell\|$
 quantum fluct.
 gaussian

expansion as Eq. (1) \Rightarrow Master eq. in co-moving frame, describes quantum fluctuations

$$\Rightarrow (4) \quad \dot{\alpha}_\ell \approx -\frac{i}{\hbar} [H_\alpha^{(\alpha)}, \rho_\alpha] + 2 \sum_{\ell=1}^N [K_\ell \mathcal{D}(\tilde{\alpha}_\ell^+) + 4K_2 |\alpha_\ell|^2 \mathcal{D}(\tilde{\alpha}_\ell)]$$

coherent dyn.

$$H_\alpha^{(\alpha)}(t) = \hbar \sum_{\ell=1}^N (iK_2 (\alpha_\ell^*)^2 \tilde{\alpha}_\ell^2 - iK_2 \alpha_\ell^2 (\tilde{\alpha}_\ell^+)^2 + \sum_{r=1}^N K_{\ell r} (\alpha_\ell^+ \tilde{\alpha}_{\ell+r} + \tilde{\alpha}_\ell \alpha_{\ell+r}^+))$$

$$+ \hbar \sum_{\ell=1}^N (-i[\dot{\alpha}_\ell \tilde{\alpha}_\ell^+ - \dot{\alpha}_\ell^* \tilde{\alpha}_\ell] + iK_1 \alpha_\ell \tilde{\alpha}_\ell^+ - iK_1 \alpha_\ell^* \tilde{\alpha}_\ell)$$

$$+ \hbar \sum_{\ell=1}^N (2iK_2 \alpha_\ell (\alpha_\ell^*)^2 \tilde{\alpha}_\ell^2 - 2iK_2 \alpha_\ell^* \alpha_\ell^2 \tilde{\alpha}_\ell^+)$$

$$+ \hbar \sum_{r=1}^N K_{\ell r} (\alpha_{\ell+r} \tilde{\alpha}_\ell^+ + \alpha_{\ell+r}^* \tilde{\alpha}_\ell + \alpha_\ell^* \alpha_{\ell+r} + \alpha_\ell \alpha_{\ell+r}^+)$$

semiclass. trajectory (\Rightarrow lin. terms in Master eq. vanish):

$$(5) \quad \dot{\alpha}_\ell(t) = \alpha_\ell(t) (K_1 - 2K_2 |\alpha_\ell(t)|^2) - i \sum_{\substack{s=1 \\ s \neq \ell}}^N K_{\ell s} \alpha_s(t)$$

$$(4) \Rightarrow \frac{d}{dt} \langle \tilde{\alpha}_i \rangle \equiv \hbar [\tilde{\alpha}_i, \dot{\alpha}_\ell(t)] = K_1 \langle \tilde{\alpha}_i \rangle - 4K_2 |\alpha_i|^2 \langle \tilde{\alpha}_i \rangle$$

$$- 2K_2 \alpha_i^2 \langle \tilde{\alpha}_i^+ \rangle$$

$$- i \sum_{\substack{s=1 \\ s \neq i}}^N K_{is} \langle \tilde{\alpha}_s \rangle$$