6.4. Maxwell-Gleichungen

6.4.1. Homogene Maxwell-Gl.

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -cB^3 & cB^2 \\ E^2 & cB^3 & 0 & -cB^1 \\ E^3 & -cB^2 & cB^1 & 0 \end{pmatrix}$$

$\phi^\nu = (\phi, c\underline{A})$

$j^\mu = (c\varrho, \underline{j})$

(1) $\nabla \cdot \underline{B} = 0 = \partial_1 B^1 + \partial_2 B^2 + \partial_3 B^3$

$$\partial_1 F^{32} + \partial_2 F^{13} + \partial_3 F^{21} = 0$$

mit $\partial_1 = -\partial^1$, ... $F^{32} = -F^{23}$, ... folgt

total antisymm. Tensor

$$\boxed{\underset{1\,23}{\partial^1 F^{23}} + \underset{2\,31}{\partial^2 F^{31}} + \underset{3\,12}{\partial^3 F^{12}} = 0}$$

zykl. (123)

(2) $\nabla \times \underline{E} = -\dot{\underline{B}}$

1. Komponente $\partial_2 E^3 - \partial_3 E^2 + \frac{\partial}{\partial t} B^1 = 0$

$$\partial_2 F^{30} - \partial_3 F^{20} + \partial_0 F^{32} = 0$$

mit $\partial_0 = \partial^0$; $\partial_2 = -\partial^2$, $\partial_3 = -\partial^3$

$F^{32} = -F^{23}$ ...

$$\boxed{\underset{0\,23}{\partial^0 F^{23}} + \underset{2\,30}{\partial^2 F^{30}} + \underset{3\,02}{\partial^3 F^{02}} = 0}$$

zykl. (023)

analog ergeben die anderen Komponenten

$$\boxed{\begin{aligned} \partial^0 F^{13} + \partial^3 F^{01} + \partial^1 F^{30} &= 0 \\ \partial^0 F^{12} + \partial^1 F^{20} + \partial^2 F^{01} &= 0 \end{aligned}}$$

zykl. (013)

zykl. (012)

$\left(1\,2\,3\,4 \quad \overset{\text{ungerade}}{4\,2\,3\,1} \quad \overset{\text{gerade}}{4\,1\,3\,2} \quad \overset{\text{ungerade}}{4\,1\,2\,3} \right)$

Zusammenfassung der homog. Maxwell-Gleichungen

$$\Rightarrow \quad \boxed{\varepsilon_{\mu\nu\lambda\kappa}\, \partial^\nu F^{\lambda\kappa} = 0}$$

oder

$$\boxed{\varepsilon^{\mu\nu\lambda\kappa}\, \partial_\nu F_{\lambda\kappa} = 0}$$

„4-Rotation"

mit $\varepsilon^{\mu\nu\lambda\kappa} = \begin{cases} 1 & \text{wenn } (\mu\nu\lambda\kappa) = \text{gerade Permutation} \\ -1 & \text{wenn } (\mu\nu\lambda\kappa) = \text{ungerade ...} \\ 0 & \text{sonst} \end{cases}$

4-dim Levi-Civita-Tensor

(analog zum 3 dim. : $\varepsilon_{ik\ell}\, \partial^k a^\ell = (\underline{\nabla} \times \underline{a})_i$)

Bemerkung: (i) $\varepsilon^{\mu\nu\lambda\kappa}$ ist vollständig antisymmetrisch

(ii) $\varepsilon'^{\mu\nu\lambda\kappa} = U^\mu{}_\pi\, U^\nu{}_\varrho\, U^\lambda{}_\sigma\, U^\kappa{}_\tau\, \varepsilon^{\pi\varrho\sigma\tau}$

Lorentz Transformation

$$= \begin{vmatrix} U^\mu{}_0 & U^\mu{}_1 & U^\mu{}_2 & U^\mu{}_3 \\ U^\nu{}_0 & U^\nu{}_1 & U^\nu{}_2 & U^\nu{}_3 \\ U^\lambda{}_0 & U^\lambda{}_1 & U^\lambda{}_2 & U^\lambda{}_3 \\ U^\kappa{}_0 & U^\kappa{}_1 & U^\kappa{}_2 & U^\kappa{}_3 \end{vmatrix} = \underbrace{(\det U)}_{\pm 1}\, \varepsilon^{\mu\nu\lambda\kappa}$$

Damit $\varepsilon'^{\mu\nu\lambda\kappa} = \varepsilon^{\mu\nu\lambda\kappa}$ ist, muss vereinbart werden dass

$$\varepsilon'^{\mu\nu\lambda\kappa} = \underbrace{(\det U)}_{\pm 1} U^{\mu}{}_{\pi} U^{\nu}{}_{\varsigma} U^{\lambda}{}_{\sigma} U^{\kappa}{}_{\tau} \varepsilon^{\pi\varsigma\sigma\tau} .$$

$\Rightarrow \varepsilon^{\mu\nu\lambda\kappa}$ ist ein Pseudotensor.

Verallgemeinerung des 3-dim Falles ε^{ikl} mit $(\underline{\nabla} \times \underline{a})_i = \varepsilon^{ikl} \partial_k a_l$.

Pseudotensor

6.4.2. Inhomogene Maxwell-Gl. (im Vakuum)

(3) $\varepsilon_0 \nabla \cdot \underline{E} = \varrho$

$$\partial_1 E^1 + \partial_2 E^2 + \partial_3 E^3 = \frac{1}{\varepsilon_0 c} c\varrho$$

$$\partial_1 F^{10} + \partial_2 F^{20} + \partial_3 F^{30} = \frac{1}{\varepsilon_0 c} j^0 \quad \Rightarrow \quad \boxed{\partial_\mu F^{\mu 0} = \frac{1}{\varepsilon_0 c} j^0} \quad , \text{ da } \partial_0 F^{00} \equiv 0$$

$\mu = 0,1,2,3$

(4) $\underline{\nabla} \times \underline{B} = \mu_0 \underline{j} + \varepsilon_0 \mu_0 \frac{\partial}{\partial t} \underline{E}$

1. Komponente:

$$\partial_2 B^3 - \partial_3 B^2 = \mu_0 j^1 + \varepsilon_0 \mu_0 \frac{\partial}{\partial t} E^1 \qquad \text{mit } \mu_0 c = \frac{1}{\varepsilon_0 c} :$$

$$\partial_2 F^{21} - \partial_3 F^{13} = \frac{1}{\varepsilon_0 c} j^1 + \partial_0 F^{10}$$

$$\partial_0 F^{01} + \partial_2 F^{21} + \partial_3 F^{31} = \frac{1}{\varepsilon_0 c} j^1 \quad \Rightarrow \quad \boxed{\partial_\mu F^{\mu 1} = \frac{1}{\varepsilon_0 c} j^1} \quad , \text{ da } \partial_1 F^{11} = 0$$

Zusammengefasst: Inhomog. Maxwell-Gl.: analog für 2. und 3. Komponente.

$$\boxed{\partial_\mu F^{\mu\nu} = \frac{1}{\varepsilon_0 c} j^\nu} \qquad (\text{"4-Divergenz"})$$

Bemerkungen:

(i) Die homogenen Maxwell-Gl. sind durch Potenzialansatz $F_{\lambda k} = \partial_\lambda \phi_k - \partial_k \phi_\lambda$ (*)

automatisch erfüllt:

$$\varepsilon^{\mu\nu\lambda k} \partial_\nu F_{\lambda k} \overset{(*)}{=} \underbrace{\varepsilon^{\mu\nu\lambda k} \partial_\nu \partial_\lambda \phi_k}_{} - \underbrace{\varepsilon^{\mu\nu\lambda k} \partial_\nu \partial_k \phi_\lambda}_{=0}$$

ε ist total antisymm.

$(\nu \leftrightarrow \lambda)$

$$\varepsilon^{\mu\nu\lambda k} \partial_\nu \partial_\lambda \phi_k = - \varepsilon^{\mu\lambda\nu k} \partial_\nu \partial_\lambda \phi_k$$

(da in $\varepsilon^{\mu\nu\lambda k} \partial_\nu \partial_\lambda \phi_k$ über alle Perm. summiert wird)

$\Downarrow$ $= 0$

$= 0$ ∎

(ii) Aus den inhom. Maxwell-Gl. sollte die Wellengleichung für die Potenziale folgen.

$$\partial_\mu F^{\mu\nu} = \frac{1}{\varepsilon_0 c} j^\nu = \partial_\mu \partial^\mu \phi^\nu - \partial_\mu \partial^\nu \phi^\mu$$

mit Lorentz-Eichung $\quad \partial_\mu \phi^\mu = 0 \quad$ folgt

$$\partial_\mu \partial^\nu \phi^\mu = \partial^\nu \partial_\mu \phi^\mu = 0$$

$$\Rightarrow \boxed{\partial_\mu \partial^\mu \phi^\nu = \frac{1}{\varepsilon_0 c} j^\nu}$$

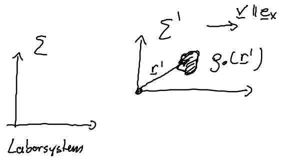
inhomogene Wellengleichung $\quad \left[\begin{array}{l} \Box \phi = -\frac{1}{\varepsilon_0}\rho \\ \Box \underline{A} = -\mu_0 \underline{j} \end{array}\right]$

(iii) Die Maxwell-gleichungen

$$\boxed{\begin{array}{l} \varepsilon^{\mu\nu\lambda\kappa} \partial_\nu F_{\lambda\kappa} = 0 \\ \partial_\mu F^{\mu\nu} = \frac{1}{\varepsilon_0 c} j^\nu \end{array}}$$

sind Lorentz-kovariant, weil sie durch 4-er Vektoren ausgedrückt werden können.

Bsp. Feld einer gleichförmig bewegten Ladungsverteilung



Strom und Felder in Σ':

- 4-er Strom in Σ' $\quad j'^\mu = \begin{cases} c\rho_0(\underline{r}') & \mu = 0 \\ 0 & \mu = 1,2,3 \end{cases}$

- Potenzial in Σ' $\quad \phi'(\underline{r}') = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho_0(\underline{r}'')}{|\underline{r}' - \underline{r}''|} d^3 r''$ (in Σ'), $\quad \underline{A}'(\underline{r}', t) = 0$

$$\phi'^\mu(\underline{r}', t') = \begin{pmatrix} \phi'(\underline{r}') \\ c A'(\underline{r}', t') \end{pmatrix} = \begin{pmatrix} \phi'(\underline{r}') \\ 0 \end{pmatrix}$$

Eichung? $\quad \partial'_\mu \phi'^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t'}, \underline{\nabla}'\right) \begin{pmatrix} \phi'(\underline{r}') \\ c\underline{A}' \end{pmatrix} = 0$

$\Rightarrow$ Lorentz-Eichung

- Felder in Σ'

$$\underline{E}'(\underline{r}', t) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho_0(\underline{r}'')(\underline{r}' - \underline{r}'')}{|\underline{r}' - \underline{r}''|^3} d^3 r'' \quad , \quad \underline{B}'(\underline{r}', t') = 0$$

(VL vom 16.10)

Strom & Felder im Laborsystem Σ:

- $j^\lambda = U^{-1\,\lambda}{}_\mu \, j'^\mu \quad$ (Rücktrafo)

$\quad = U_\mu{}^\lambda \, j'^\mu \quad\quad U_\mu{}^\lambda = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \quad$ (ohne y & z Komp.)

$$\underbrace{\begin{pmatrix} c\rho \\ j_x \end{pmatrix}}_{j^\lambda} = j^\lambda = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \underbrace{\begin{pmatrix} c\rho_0(\underline{r}') \\ 0 \end{pmatrix}}_{j'^\mu} \Rightarrow \begin{array}{ll} c\rho(\underline{r}, t) = \gamma c \rho_0(\underline{r}') & (1) \\ j_x(\underline{r}, t) = \beta\gamma c \rho_0(\underline{r}') & (2) \end{array}$$

Transformation der Koordinaten

$$x'^{\lambda} = U^{\lambda}{}_{\mu} x^{\mu} \qquad x^{\mu} \text{ (Hin-Trafo)}$$

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$$

$$\overset{①}{\Rightarrow} \rho(\underline{r},t) = \gamma \rho_0(\gamma(x-vt), y, z)$$

$$\overset{②}{\Rightarrow} \underline{j}(\underline{r},t) = \underline{v} \cdot \rho(\underline{r},t)$$

⇘ Ladungsverteilung erscheint in Σ in x gestaucht

Ladungserhaltung erfüllt? $\underset{\text{in } \Sigma'}{\int \rho_0(\underline{r}')d^3r'} = \underset{\text{in } \Sigma}{\int \rho(\underline{r},t)\, d^3r}$

$$= \gamma \int \rho_0(\gamma(x-vt), y, z)\, d^3r$$

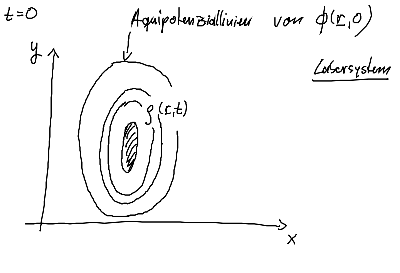
ist erfüllt!

- Potenziale in Σ

$$A^{\mu} = \begin{pmatrix} \phi(\underline{r},t) \\ cA_x(\underline{r},t) \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} \phi'(r') \\ 0 \end{pmatrix}$$

$$\phi(\underline{r},t) = \frac{1}{4\pi\varepsilon_0}\gamma \int \frac{\rho_0(r'')}{\sqrt{(\gamma x - \beta\gamma ct - x'')^2 + (y-y'')^2 + (z-z'')^2}}\, d^3r''$$

$$cA_x(\underline{r},t) = \beta\phi(\underline{r},t)$$
$$A_y = 0$$
$$A_z = 0$$



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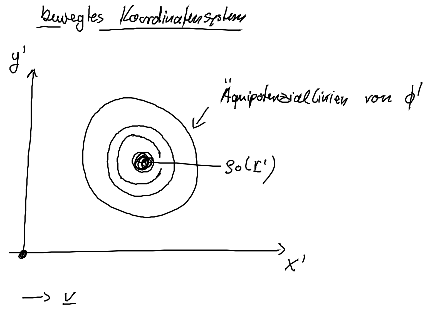

$$\underline{A}' = 0$$