

### English Summary:

#### Maxwell's equations in vacuum

$\begin{aligned} \nabla \times \underline{E} + \dot{\underline{B}} &= 0 \\ \nabla \cdot \underline{B} &= 0 \\ \nabla \cdot \underline{D} &= \rho \\ \nabla \times \underline{H} - \dot{\underline{D}} &= \underline{j} \end{aligned}$	$\begin{aligned} \oint_{\partial F} \underline{E} \cdot d\underline{s} &= - \frac{\partial}{\partial t} \underbrace{\int_F \underline{B} \cdot d\underline{f}}_{\Phi} \\ \oint_{\partial V} \underline{B} \cdot d\underline{f} &= 0 \\ \oint_{\partial V} \underline{D} \cdot d\underline{f} &= Q \\ \oint_{\partial F} \underline{H} \cdot d\underline{s} &= \int_F \underline{j} \cdot d\underline{f} + \dot{I} \end{aligned}$	<p><math>\Phi</math> magn. flux</p> <p>integral form</p>
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$\underline{D}(\underline{r}, t) = \epsilon_0 \underline{E}(\underline{r}, t)$  dielectric displacement

$\underline{H}(\underline{r}, t) = \frac{1}{\mu_0} \underline{B}(\underline{r}, t)$  magnetic field

### 3.3 TCP - Invarianz

Invarianzeigenschaften der Maxwell-ges.:

- Lorentz-Invarianz (s. Kap. 6)
- TCP-Invarianz

außerdem:

- linear in  $\underline{E}, \underline{B}, \underline{D}, \underline{H}$  (Superpositionsprinzip!)
- 1. Ordnung in  $t$  (Kausalitätsprinzip:  $\underline{E}, \underline{B}, \underline{D}, \underline{H}$  zur Zeit  $t=0$  soll den Zustand für  $t>0$  vollständig festlegen!)

Zeitumkehr  $T: t \rightarrow t' = -t$

Ladungsumkehr  $C: Q \rightarrow Q' = -Q$

Paritätsumkehr  $P: \underline{r} \rightarrow \underline{r}' = -\underline{r}$

#### (i) Zeitumkehr-Transformation

$T_g := \{ T\text{-invariante Observable } A: TA = A \}$   
 $= \{ \underline{r}, d\underline{r}, \underline{a} := \frac{d^2 \underline{r}}{dt^2}, m, q, \rho = \lim_{\Delta V \rightarrow 0} \frac{\Delta Q}{\Delta V}, \underline{F} = m \underline{a}, \underline{E} = \frac{1}{q} \underline{F}, \text{ Potential } \phi, \dots \}$   
 „gerade“ unter  $T$

$T_u := \{ A: TA = -A \}$  „ungerade“ unter  $T$   
 $= \{ \underline{v} = \frac{d\underline{r}}{dt}, \underline{j} = \rho \underline{v}, \underline{B}, \underline{A}, \dots \}$

denn:  $\underline{F} = q \underline{v} \times \underline{B}, \underline{B} = \nabla \times \underline{A}$   
 $\begin{matrix} \uparrow & \uparrow & \uparrow \\ T_g & T_g T_u & T_u T_g \end{matrix}$

⇒ T-Invarianz der Maxwell-gln.:

$$T: \{ \nabla \times \underline{E} + \dot{\underline{B}} = 0 \} \rightarrow \{ \nabla \times \underline{E} + \dot{\underline{B}} = 0 \}$$

$$T: \{ \nabla \cdot \underline{B} = 0 \} \rightarrow \{ -\nabla \cdot \underline{B} = 0 \}$$

$$T: \{ \epsilon_0 \nabla \cdot \underline{E} = \rho \} \rightarrow \{ \epsilon_0 \nabla \cdot \underline{E} = \rho \}$$

$$T: \{ \nabla \times \underline{H} - \dot{\underline{D}} = \underline{j} \} \rightarrow \{ -\nabla \times \underline{H} + \dot{\underline{D}} = -\underline{j} \}$$

Kontinuitätsgl.:

$$T: \{ \frac{\partial \rho}{\partial t} + \nabla \cdot \underline{j} = 0 \} \rightarrow \{ -\frac{\partial \rho}{\partial t} - \nabla \cdot \underline{j} = 0 \}$$

Die Maxwell-gln. sind forminvariant!

(ii) Ladungsumkehr (-konjugation)

$$C_q = \{ \underline{E}, \underline{m}, \underline{r}, \underline{v}, \underline{a}, \dots \} \text{ gerade unter } C$$

$$C_u = \{ \underline{E} = \frac{1}{q} \underline{F}, \underline{B}, \underline{s}, \underline{j}, \dots \} \text{ ungerade unter } C$$

da  $\underline{F} = q \underline{v} \times \underline{B}$

⇒ C-Invarianz der Maxwell-gln.:

$$C: \{ \nabla \times \underline{E} + \dot{\underline{B}} = 0 \} \rightarrow \{ -\nabla \times \underline{E} - \dot{\underline{B}} = 0 \}$$

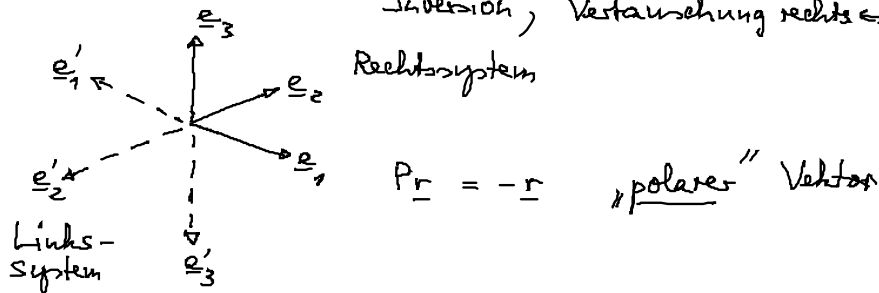
$$C: \{ \nabla \cdot \underline{B} = 0 \} \rightarrow \{ -\nabla \cdot \underline{B} = 0 \}$$

$$C: \{ \epsilon_0 \nabla \cdot \underline{E} = \rho \} \rightarrow \{ -\epsilon_0 \nabla \cdot \underline{E} = -\rho \}$$

$$C: \{ \nabla \times \underline{H} - \dot{\underline{D}} = \underline{j} \} \rightarrow \{ -\nabla \times \underline{H} + \dot{\underline{D}} = -\underline{j} \}$$

$$C: \{ \frac{\partial \rho}{\partial t} + \nabla \cdot \underline{j} = 0 \} \rightarrow \{ -\frac{\partial \rho}{\partial t} - \nabla \cdot \underline{j} = 0 \}$$

(iii) Paritätsumkehr (Räumliche Spiegelung am Ursprung, Inversion, Vertauschung rechts ↔ links)



aber  $\underline{P} (\underline{a} \times \underline{b}) = (-\underline{a}) \times (-\underline{b}) = \underline{a} \times \underline{b}$  „axiales“ Vektor  
P-invariant (Pseudovektor)

Seien  $\underline{a}, \underline{b}$  polar  $\Rightarrow \underline{a} \times \underline{\omega}$  polar  
 $\underline{\omega}, \underline{\sigma}$  axial  $\underline{a} \times \underline{b}, \underline{\omega} \times \underline{\sigma}$  axial  
 $\underline{a} \cdot \underline{b}$  skalar  $P(\underline{a} \cdot \underline{b}) = \underline{a} \cdot \underline{b}$   
 $\underline{a} \cdot \underline{\omega}$  pseudoskalar  $P(\underline{a} \cdot \underline{\omega}) = -\underline{a} \cdot \underline{\omega}$

$P_g = \{ \text{skalare } m, q; \text{ axiale Vektoren } \underline{B} \}$  gerade

$$\underline{F} = q \underline{v} \times \underline{B}$$

$$\begin{matrix} \uparrow & \uparrow & \uparrow & \uparrow \\ P_u & P_g & P_u & P_g \end{matrix}$$

$P_u = \{ \text{polare Vektoren } \underline{r}, \underline{d}\underline{r}, \underline{v}, \underline{a}, \underline{F}, \underline{E} = \frac{1}{q} \underline{F}, \underline{j} = q \underline{v}, \underline{A};$   
Pseudoskalar  $\underline{\nabla} \cdot \underline{B} \}$   $\underline{B} = \underline{\nabla} \times \underline{A}$   
 $\begin{matrix} u & g \\ g & u & u \end{matrix}$

P-Invarianz der Maxwell-gln.:

$$P: \{ \underline{\nabla} \times \underline{E} + \underline{\dot{B}} = 0 \} \rightarrow \{ \underline{\nabla} \times \underline{E} + \underline{\dot{B}} = 0 \}$$

$$P: \{ \underline{\nabla} \cdot \underline{B} = 0 \} \rightarrow \{ -\underline{\nabla} \cdot \underline{B} = 0 \} !$$

$$P: \{ \epsilon_0 \underline{\nabla} \cdot \underline{E} = \rho \} \rightarrow \{ \epsilon_0 \underline{\nabla} \cdot \underline{E} = \rho \}$$

$$P: \{ \underline{\nabla} \times \underline{H} - \underline{\dot{D}} = \underline{j} \} \rightarrow \{ -\underline{\nabla} \times \underline{H} + \underline{\dot{D}} = \underline{j} \}$$

$$P: \{ \frac{\partial}{\partial t} \rho + \underline{\nabla} \cdot \underline{j} = 0 \} \rightarrow \{ \frac{\partial}{\partial t} \rho + \underline{\nabla} \cdot \underline{j} = 0 \}$$

NB: Schwache WW verletzt die Paritäts-erhaltung (Pauli 1957)  
(P, PC, C, T verletzt, aber PCT ist erhalten)

Lorentzkraft  $\underline{F} = q(\underline{E} + \underline{v} \times \underline{B})$

soll aus einem Extremal-Prinzip (Hamilton'sches Prinzip) ableitbar sein.

$\Rightarrow$  Zus.hang zwischen  $\underline{E}, \underline{B}$  und  $\phi, \underline{A}$

Suche Lagrange-Fkt.  $L(\underline{r}, \underline{v}, t)$ , so dass Lagrange-ges.

$$\frac{d}{dt} \frac{\partial L}{\partial \underline{v}_k} - \frac{\partial L}{\partial x_k} = 0$$

$$\text{ergibt: } m \underline{\ddot{r}} = q [\underline{E}(\underline{r}, t) + \underline{v} \times \underline{B}(\underline{r}, t)] \quad (*)$$

$$\text{Lagrange-Fkt. } \boxed{L = \frac{m}{2} \underline{v}^2 + q [\underline{v} \cdot \underline{A} - \phi(\underline{r}, t)]}$$

Tatsächlich gilt

$$p_k = \frac{\partial L}{\partial v_k} = m v_k + q A_k(r, t) = \text{„kanon. Impuls“}$$

kinet.  
Impuls

$$\begin{aligned} \frac{d}{dt} \frac{\partial L}{\partial v_k} &= m \ddot{x}_k + q \frac{d}{dt} A_k(r, t) \\ &\quad \text{totale Zeitableit, längs einer Bahn } \underline{r}(t) \\ &= m \ddot{x}_k + q \left( \frac{\partial}{\partial t} A_k + \sum_l \frac{\partial A_k}{\partial x_l} \dot{x}_l \right) \\ &= m \ddot{x}_k + q \left( \frac{\partial}{\partial t} + \underline{v} \cdot \nabla \right) A_k \end{aligned}$$

$$\frac{\partial L}{\partial x_k} = q \left[ \frac{\partial}{\partial x_k} (\underline{v} \cdot \underline{A}) - \frac{\partial}{\partial x_k} \phi \right]$$

$$\Rightarrow 0 = \frac{d}{dt} \frac{\partial L}{\partial v_k} - \frac{\partial L}{\partial x_k} = m \ddot{x}_k + q \frac{\partial}{\partial t} A_k + q \left[ (\underline{v} \cdot \nabla) A_k - \frac{\partial}{\partial x_k} (\underline{v} \cdot \underline{A}) \right] + q \frac{\partial}{\partial x_k} \phi$$

$- [\underline{v} \times (\nabla \times \underline{A})]_k$

$$0 = m \ddot{\underline{r}} + q \left[ \frac{\partial}{\partial t} \underline{A} + \nabla \phi - \underline{v} \times (\nabla \times \underline{A}) \right]$$

Vergleich mit Lorentzkraft  $\underline{F}$ :

$$\begin{aligned} \underline{E}(r, t) &= - \frac{\partial}{\partial t} \underline{A}(r, t) - \nabla \phi(r, t) \\ \underline{B}(r, t) &= \nabla \times \underline{A}(r, t) \end{aligned}$$

damit sind die homog. Maxwell-Gln. automatisch erfüllt!

$$\nabla \cdot \underline{B} = \nabla \cdot (\nabla \times \underline{A}) \equiv 0$$

$$\nabla \times \underline{E} = - \frac{\partial}{\partial t} \underbrace{\nabla \times \underline{A}}_{\underline{B}} - \underbrace{\nabla \times (\nabla \phi)}_{\equiv 0}$$

### 3.4 Energiebilanz

Die Maxwell-Gln. enthalten die Kontingf. für  $\rho$ :

$$\frac{\partial}{\partial t} \rho + \nabla \cdot \underline{j} = 0 \quad (\text{Ladungserhaltung})$$

Frage: weitere Erhaltungssätze für extensive phys. Obs., z.B. Energie, Impuls, Drehimpuls?  
(Ext.: additiv bei Systemzus.setzung)  
Intensiv: z.B. Temp.

Energietransport durch das el. magn. Feld:

$$\nabla \times \underline{E} + \frac{\partial \underline{B}}{\partial t} = 0 \quad | \cdot \underline{H}$$

$$\nabla \times \underline{H} - \frac{\partial \underline{D}}{\partial t} = \underline{j} \quad | \cdot \underline{E}$$

$$\rightarrow \underbrace{\underline{H} \cdot (\nabla \times \underline{E}) - \underline{E} \cdot (\nabla \times \underline{H})}_{\nabla \cdot (\underline{E} \times \underline{H})} + \underbrace{\underline{H} \frac{\partial \underline{B}}{\partial t}}_{\frac{1}{\mu_0} \frac{\partial \underline{B}^2}{\partial t}} + \underbrace{\underline{E} \frac{\partial \underline{D}}{\partial t}}_{\epsilon_0 \frac{\partial \underline{E}^2}{\partial t}} = -\underline{j} \cdot \underline{E}$$

$$\frac{1}{2} \frac{\partial \underline{B}^2}{\partial t} \quad \frac{1}{2} \frac{\partial \underline{E}^2}{\partial t}$$

$$\Rightarrow \boxed{\frac{\partial w}{\partial t} + \nabla \cdot \underline{S} = -\underline{j} \cdot \underline{E}} \quad \text{Kontin. gl.}$$

= Bilanzgl. für Energietransport

$$w := \frac{\epsilon_0}{2} \underline{E}^2 + \frac{1}{2\mu_0} \underline{B}^2 = \frac{1}{2} (\underline{E} \cdot \underline{D} + \underline{B} \cdot \underline{H}) \quad \underline{\text{Energiedichte}}$$

$$\underline{S} := \underline{E} \times \underline{H} \quad \underline{\text{Energiestromdichte des el. magn. Feldes}}$$

( Poynting - Vektor )

$$\sigma = -\underline{j} \cdot \underline{E} \quad \text{Quelldichte der Feldenergie}$$

( Leistungsdichte )

$$\left. \begin{array}{l} \underline{j} \cdot \underline{E} > 0 \quad \text{Abnahme} \\ \underline{j} \cdot \underline{E} < 0 \quad \text{Zunahme} \end{array} \right\} \text{des Feldenergie}$$