

English Summary:

Transformation of current and fields

4-current density $j^\mu := (c\rho, \underline{j})$

charge conservation $\partial_\mu j^\mu = 0$ $\partial_\mu = \frac{\partial}{\partial x^\mu}$

Lorentz invariant
d'Alembert op. $\square = -\partial_\mu \partial^\mu = \Delta - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}$

wave eq. $\partial_\mu \partial^\mu \phi^\nu = \frac{1}{\epsilon_0 c} j^\nu$

4-potential $\phi^0 := \phi$, $\phi^i := cA^i$ ($i=1,2,3$)

gauge transformation $\tilde{\phi}^\mu = \phi^\mu + \partial^\mu \varphi(x^\nu)$

scalar gauge fct.

Felder $\underline{E}, \underline{B}$

$$\underline{E} = -\nabla\phi - \frac{\partial}{\partial t}\underline{A}$$

$$\Rightarrow E^i = -\partial_i \phi - \frac{1}{c} \frac{\partial}{\partial t} cA^i = -\partial_i \phi^0 - \partial_0 \phi^i = \partial^i \phi^0 - \partial^0 \phi^i \quad (i=1,2,3)$$

$$\underline{B} = \nabla \times \underline{A}$$

$$\Rightarrow cB^1 = \partial_2 cA^3 - \partial_3 cA^2 = \partial_2 \phi^3 - \partial_3 \phi^2 = \partial^3 \phi^2 - \partial^2 \phi^3$$

zyklische Vertauschung: $cB^2 = \partial^1 \phi^3 - \partial^3 \phi^1$
 $cB^3 = \partial^2 \phi^1 - \partial^1 \phi^2$

Zusammenfassung antisymm. Feldtensor

$$F^{\mu\nu} := \partial^\mu \phi^\nu - \partial^\nu \phi^\mu, \quad F^{\mu\nu} = -F^{\nu\mu} \quad (\mu, \nu = 0, 1, 2, 3)$$

Wegen der Antisymm. hat $F^{\mu\nu}$ nur 6 unabhängige Komponenten:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -cB^3 & cB^2 \\ E^2 & cB^3 & 0 & -cB^1 \\ E^3 & -cB^2 & cB^1 & 0 \end{pmatrix}$$

Lorentz-Transform der Felder

Als Tensor 2. Stufe transformiert sich $F^{\mu\nu}$ wie

$$F'^{\mu\nu} = U^\mu_\alpha U^\nu_\kappa F^{\alpha\kappa}, \quad U^\mu_\nu = \begin{pmatrix} \gamma & -\beta\gamma & 0 \\ -\beta\gamma & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Im einzelnen:

$$\begin{aligned} E'^1 &= F'^{10} = U^1_\alpha U^0_\kappa F^{\alpha\kappa} = -\beta\gamma U^0_\kappa F^{0\kappa} + \gamma U^0_\kappa F^{1\kappa} \\ &= (\beta\gamma)^2 F^{01} + (\gamma)^2 F^{10} \\ &= \underbrace{\gamma^2(1-\beta^2)}_1 F^{10} = E^1 \end{aligned}$$

$$\begin{aligned} E'^2 &= F'^{120} = U^2_\alpha U^0_\kappa F^{\alpha\kappa} = U^0_\kappa F^{2\kappa} = \gamma F^{20} - \beta\gamma F^{21} \\ &= \gamma(E^2 - vB^3) \end{aligned}$$

$$\begin{aligned} E'^3 &= F'^{130} = U^0_\alpha U^3_\kappa F^{\alpha\kappa} = \gamma F^{30} - \beta\gamma F^{31} \\ &= \gamma(E^3 + vB^2) \end{aligned}$$

$$B'^1 = \frac{1}{c} F'^{32} = \frac{1}{c} U^3_\alpha U^2_\kappa F^{\alpha\kappa} = \frac{1}{c} F^{32} = B^1$$

$$\begin{aligned} B'^2 &= \frac{1}{c} F'^{13} = \frac{1}{c} U^1_\alpha U^3_\kappa F^{\alpha\kappa} = \frac{1}{c} U^1_\alpha F^{\alpha 3} = -\frac{\beta\gamma}{c} F^{03} + \frac{\gamma}{c} F^{13} \\ &= \gamma(B^2 + \frac{v}{c^2} E^3) \end{aligned}$$

$$B'^3 = \gamma(B^3 - \frac{v}{c^2} E^2)$$

Zusammenfassung:

$E'^1 = E^1$	$B'^1 = B^1$
$E'^2 = \gamma(E^2 - vB^3)$	$B'^2 = \gamma(B^2 + \frac{v}{c^2} E^3)$
$E'^3 = \gamma(E^3 + vB^2)$	$B'^3 = \gamma(B^3 - \frac{v}{c^2} E^2)$

Elektr. u. magnet. Felder werden beim Übergang zwischen verschiedenen Inertialsystemen in einander transformiert.

Umrechnung

$$\tilde{\phi}^\mu = \phi^\mu + \partial^\mu \varphi$$

$$\begin{aligned} \Rightarrow \tilde{F}^{\mu\nu} &= \partial^\mu \tilde{\phi}^\nu - \partial^\nu \tilde{\phi}^\mu = \partial^\mu (\phi^\nu + \partial^\nu \varphi) - \partial^\nu (\phi^\mu + \partial^\mu \varphi) \\ &= \underbrace{\partial^\mu \phi^\nu - \partial^\nu \phi^\mu}_{F^{\mu\nu}} + \underbrace{\partial^\mu \partial^\nu \varphi - \partial^\nu \partial^\mu \varphi}_0 = F^{\mu\nu} \end{aligned}$$

also $F^{\mu\nu}$ lichinvariant!

Homogene Maxwellgl.

$$(1) \nabla \cdot \underline{B} = \partial_1 B^1 + \partial_2 B^2 + \partial_3 B^3 = 0$$

$$\Rightarrow \partial_1 F^{32} + \partial_2 F^{13} + \partial_3 F^{21} = 0$$

mit $\partial_1 = -\partial^1, \dots, F^{32} = -F^{23}, \dots$ folgt:

$$\boxed{\partial^1 F^{23} + \partial^2 F^{31} + \partial^3 F^{12} = 0} \quad \text{zykl. (123)}$$

123 231 312

$$(2) \nabla \times \underline{E} + \frac{\partial}{\partial t} \underline{B} = 0$$

$$1. \text{ Komp. : } \partial_2 E^3 - \partial_3 E^2 + \frac{\partial}{\partial t} B^1 = 0$$

$$\Rightarrow \partial_2 F^{30} - \partial_3 F^{20} + \partial_0 F^{32} = 0$$

mit $\partial_0 = \partial^0, \partial_2 = -\partial^2, \partial_3 = -\partial^3, F^{32} = -F^{23}, F^{20} = -F^{02}$:

$$\boxed{\partial^0 F^{23} + \partial^2 F^{30} + \partial^3 F^{02} = 0} \quad \text{zykl. (023)}$$

zykl. Permutation $1 \rightarrow 2 \rightarrow 3 \rightarrow 1$ und $F^{\mu\nu} = -F^{\nu\mu}$

$$\boxed{\partial^0 F^{13} + \partial^3 F^{01} + \partial^1 F^{30} = 0} \quad \text{zykl. (013)}$$

$$\boxed{\partial^0 F^{12} + \partial^1 F^{20} + \partial^2 F^{01} = 0} \quad \text{zykl. (012)}$$

Zusammenfassung der homog. Maxwellgl.:

$$\boxed{\epsilon_{\mu\nu\lambda\kappa} \partial^\nu F^{\lambda\kappa} = 0}$$

oder

$$\boxed{\epsilon^{\mu\nu\lambda\kappa} \partial_\nu F_{\lambda\kappa} = 0}$$

"4-Rotation"

mit

$$\epsilon^{\mu\nu\lambda\kappa} = \begin{cases} 1 & \text{wenn } (\mu\nu\lambda\kappa) = \text{gerade Permutation von } (0123) \\ -1 & \text{wenn } (\mu\nu\lambda\kappa) = \text{ungerade Perm. v. } (0123) \\ 0 & \text{sonst} \end{cases}$$

(4-dim. Levi-Civita-Tensor)

3-dim. Levi-Civita-Tensor $\epsilon_{ikl} \partial^k a^l = (\nabla \times \underline{a})_i$

Bemerkung: (i) $\epsilon^{\mu\nu\lambda\kappa}$ ist vollständig antisymmetrisch,

(ii) $\epsilon^{\mu\nu\lambda\kappa}$ transformiert sich unter Lorentz-Transf.:

$$\epsilon^{\mu\nu\lambda\kappa} = U^\mu_\pi U^\nu_\sigma U^\lambda_\tau U^\kappa_\rho \epsilon^{\pi\sigma\rho\tau}$$

$$= \begin{vmatrix} U^0_0 & U^0_1 & U^0_2 & U^0_3 \\ U^1_0 & U^1_1 & U^1_2 & U^1_3 \\ U^2_0 & U^2_1 & U^2_2 & U^2_3 \\ U^3_0 & U^3_1 & U^3_2 & U^3_3 \end{vmatrix} = \underbrace{(\det U)}_{\pm 1} \epsilon^{\mu\nu\lambda\kappa}$$

Damit $\epsilon^{\mu\nu\lambda\kappa} = \epsilon^{\mu\nu\lambda\kappa}$ ist, muss vereinbart werden, dass

$$\epsilon^{\mu\nu\lambda\kappa} = \underbrace{(\det U)}_{\pm 1} U^\mu_\pi U^\nu_\sigma U^\lambda_\tau U^\kappa_\rho \epsilon^{\pi\sigma\rho\tau}$$

Damit ist $\epsilon^{\mu\nu\lambda\kappa}$ ein Pseudotensor

(Verallgemeinerung des 3-dim. Falles ϵ^{ikl} mit
 $(\nabla \times \underline{a})_i = \epsilon^{ikl} \partial_k a_l$
 Pseudovektor)