English Summary:

Lorentz-Lorenz local field $\underline{E}_a = \underline{E} + \frac{1}{3\varepsilon_0}\underline{P}$

macrosc. field

$\Rightarrow \underline{P} = \varepsilon_0 \chi_e \underline{E}$, $\boxed{\chi_e = \frac{n\alpha}{1-\frac{1}{3}n\alpha}}$, $\boxed{n\alpha = 3\frac{\varepsilon-1}{\varepsilon+2}}$

Clausius-Mossotti

α polarizability

n mean atom density

Wave propagation in matter

$$\boxed{\Delta \underline{E} - \frac{\varepsilon\mu}{c^2}\left(\ddot{\underline{E}} + \frac{1}{\tau}\dot{\underline{E}}\right) = 0}$$

$\tau \equiv \frac{\varepsilon_0 \varepsilon}{\sigma}$ dielectric relaxation time (damping)

$$\boxed{k^2 = \varepsilon\mu \frac{\omega^2}{c^2}\left(1 + i\frac{1}{\omega\tau}\right)}$$

dispersion relation

$k = \frac{\omega}{c}(n + i\kappa)$

refractive index, absorption coefficient

b) Dielektrische Dispersion (Ann. $\mu = 1$)

Betrachte nun zeitliche Dispersion, d.h. $\hat{\chi}(\omega)$:

$$\boxed{\hat{\underline{P}}(\omega) = \varepsilon_0 \hat{\chi}(\omega)\hat{\underline{E}}(\omega)}$$

mit $\hat{\chi}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dt\, \chi(t) e^{i\omega t}$ (dynam. el. Suszeptibilität)

Fourier-Trafo: $\underline{P}(\underline{r},t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega\, \hat{\underline{P}}(\underline{r},\omega) e^{-i\omega t}$

$$\hat{\underline{E}}(\underline{r},\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dt\, \underline{E}(\underline{r},t) e^{i\omega t}$$

$$\underline{P}(\underline{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, \varepsilon_0 \hat{\chi}(\omega) \int_{-\infty}^{\infty} dt'\, e^{i\omega(t'-t)}\, \underline{E}(\underline{r},t')$$

$\chi(t-t')$

$$= \frac{\varepsilon_0}{\sqrt{2\pi}}\int_{-\infty}^{t} dt'\, \chi(t-t')\, \underline{E}(\underline{r},t')$$

(Nachwirkungseffekt: Faltungsintegral)

NB: Kausalität verlangt $\boxed{\chi(t-t') = 0 \text{ für } t' > t}$

Aus mikroskop. Modellen folgt i.a. ein komplexes $\hat{\chi}(\omega) \in \mathbb{C}$

$\Rightarrow$ <u>komplexe diel. Funktion</u> :

$$\epsilon(\omega) = 1 + \hat{\chi}(\omega) = \epsilon'(\omega) + i\epsilon''(\omega) \quad \text{mit } \epsilon', \epsilon'' \in \mathbb{R}$$

aus $\epsilon(\omega) = 1 + \frac{1}{\sqrt{2\pi}} \int_0^\infty dt\, \chi(t) e^{i\omega t}$ folgt

$$\boxed{\epsilon^*(\omega) = \epsilon(-\omega)} \quad \Rightarrow \quad \begin{aligned} \epsilon'(\omega) &= \epsilon'(-\omega) \\ \epsilon''(\omega) &= -\epsilon''(-\omega) \end{aligned}$$

<u>monochromat. ebene Welle</u> $\underline{E}(\underline{r},t) = \underline{E}_0 e^{i(\underline{k}\cdot\underline{r} - \omega t)}$

$$\Rightarrow \boxed{k^2 = \epsilon(\omega) \frac{\omega^2}{c^2}\left(1 + i\frac{1}{\omega\tau}\right)} \qquad \tau = \frac{\epsilon_0 \epsilon(\omega)}{\sigma}$$

<u>Isolator</u> (dispersives Dielektrikum): $\boxed{k^2 \approx \epsilon(\omega) \frac{\omega^2}{c^2}}$

$k = \frac{\omega}{c}\tilde{n}$, $\quad \tilde{n}(\omega) = n(\omega) + i\varkappa(\omega)$ komplexer, frequenzabh. Brechungsindex

mit. $\quad \tilde{n}(\omega)^2 = \epsilon(\omega) = \epsilon'(\omega) + i\epsilon''(\omega)$

$$\Rightarrow \left.\begin{aligned} \epsilon'(\omega) &= n^2 - \varkappa^2 \\ \epsilon''(\omega) &= 2n\varkappa \end{aligned}\right\} \quad \boxed{\begin{aligned} \varkappa &= \\ n &= \end{aligned}\left\} \frac{1}{\sqrt{2}}\left(\sqrt{\epsilon'^2 + \epsilon''^2} \mp \epsilon'\right)^{1/2}\right.}$$

Abs. koeff. $\varkappa$
reeller Brechungsindex n

(i) <u>Absorption</u>

a) $\epsilon'' = 0 \quad \Rightarrow \quad \left\{\begin{aligned} &\text{Abs. koeff. } \varkappa = 0 \\ &\text{Brech. ind. } n = \sqrt{\epsilon'} \end{aligned}\right\}$ falls $\epsilon' > 0$ $\Rightarrow$ ungedämpfte Welle

b) $\epsilon'' > 0 \quad \Rightarrow \quad \varkappa > 0$ (gedämpfte Welle) $\Rightarrow$ Energie-dissipation

Der Frequenzbereich mit $\epsilon'' \ll \epsilon'$ heißt <u>Transparenzgebiet</u> der Substanz (wenig Absorption)

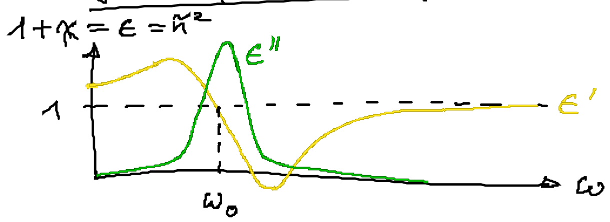
(ii) <u>Dispersion</u>

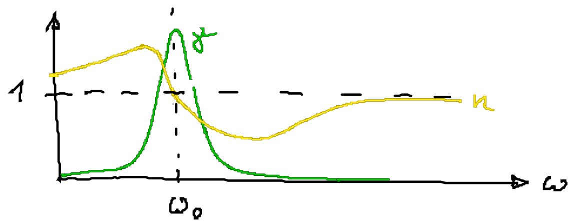
$\mathrm{Re}\, k \equiv k' = \frac{\omega}{c} n(\omega)$ nichtlin. Dispersion

$\Rightarrow$ Gruppengeschw. $v_g := \frac{d\omega}{dk'} = \frac{1}{\frac{dk'}{d\omega}} = \frac{c}{\frac{d(\omega n)}{d\omega}}$

$$\boxed{v_g = \frac{c}{n + \omega \frac{dn}{d\omega}} \neq \frac{c}{n(\omega)} = v_{ph}}$$

<u>Typ. Frequenzabhängigkeit</u> (Resonanzverh.)





normale Dispersion: $\frac{dn}{d\omega} > 0$ (stets im Transparenzgebiet)
$\epsilon'' \approx 0$, $v_g < v_{ph}$

anomale Dispersion: $\frac{dn}{d\omega} < 0$ (bei Absorption)

Ref.: Louis de Broglie: Wave propagation

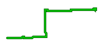
<u>Beziehungen zwischen $\epsilon'(\omega)$ und $\epsilon''(\omega)$</u>:
(Kramers–Kronig–Relationen)

- Allg. gültiger Zus.hang zwischen Dispersion $n(\omega)$ und Absorption ($\gamma(\omega)$) erlaubt z.B. Berechnung der Dispersionsbez. aus dem Absorptions-spektrum und umgekehrt!
- Folgt aus dem <u>Kausalitätsprinzip</u>!

__Beweis__ (Methode Funktionentheorie)

Für eine __kausale__ Fkt. $\chi(t)$ gilt:

$$\chi(t) = \Theta(t)\chi(t) \quad \text{mit} \quad \Theta(t) = \begin{cases} 0 & \text{für } t<0 \\ 1 & \text{für } t\geq 0 \end{cases}$$

Heaviside-Fkt. 

Fourier-Trafo:

$$\hat{\chi}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega' \, \hat{\Theta}(\omega-\omega') \, \hat{\chi}(\omega')$$

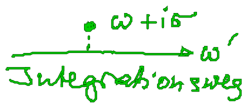
$$\hat{\Theta}(\omega) := \lim_{\sigma\to 0^+} \frac{1}{\sqrt{2\pi}} \int_0^{\infty} dt \, e^{i\omega t - \sigma t}$$

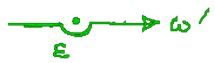
(konvergenz-erzeugender Faktor $e^{-\sigma t}$)

$$= \lim_{\sigma\to 0^+} \frac{1}{\sqrt{2\pi}} \frac{-1}{i\omega - \sigma}$$

$$\Rightarrow \hat{\chi}(\omega) = \lim_{\sigma\to 0^+} \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\omega' \, \frac{1}{\omega'-\omega-i\sigma} \, \hat{\chi}(\omega')$$

Integrand hat Pol bei $\omega' = \omega + i\sigma$



äquivalenter Integrationsweg 

Zerlegung:

$$\int_{-\infty}^{\infty} d\omega' \frac{\hat{\chi}(\omega')}{\omega'-\omega} = \lim_{\varepsilon\to 0^+} \Big[\int_{-\infty}^{\omega-\varepsilon} + \int_{\omega+\varepsilon}^{\infty} \Big] d\omega' \frac{\hat{\chi}(\omega')}{\omega'-\omega} + \underbrace{\int_{\smile} d\omega' \frac{\hat{\chi}(\omega')}{\omega'-\omega}}_{\text{Integral längs Halbkreis mit Radius } \varepsilon \text{ um Pol}}$$

$$\mathcal{P}\int_{-\infty}^{\infty} d\omega' \frac{\hat{\chi}}{\omega'-\omega} \quad \text{„\underline{Hauptwert}" (Principal value)}$$

$$\int_{\smile} d\zeta \frac{f(\zeta)}{\zeta} = f(0) \int \frac{d\zeta}{\zeta} \underset{\substack{\zeta = \varepsilon e^{i\varphi} \\ d\zeta = i\zeta \, d\varphi}}{=} f(0)\, i \int_{\pi}^{2\pi} d\varphi = i\pi f(0)$$

halbes Residuum

$$\Rightarrow \hat{\chi}(\omega) = \frac{1}{2\pi i}\, \mathcal{P}\!\int_{-\infty}^{\infty} d\omega' \, \frac{\hat{\chi}(\omega')}{\omega'-\omega} + \frac{1}{2}\hat{\chi}(\omega)$$

$$\Rightarrow \boxed{\hat{\chi}(\omega) = \frac{1}{\pi i}\, \mathcal{P}\!\int_{-\infty}^{\infty} d\omega' \, \frac{\hat{\chi}(\omega')}{\omega'-\omega}}$$

Zerlegung in Re und Im mit $\mathrm{Re}\,\hat{\chi}(\omega) = \epsilon'(\omega) - 1$

$\mathrm{Im}\,\hat{\chi}(\omega) = \epsilon''(\omega)$

$$\boxed{\begin{aligned} \epsilon'(\omega) - 1 &= \frac{1}{\pi}\, \mathcal{P}\!\int_{-\infty}^{\infty} d\omega' \, \frac{\epsilon''(\omega')}{\omega'-\omega} \\ \epsilon''(\omega) &= -\frac{1}{\pi}\, \mathcal{P}\!\int_{-\infty}^{\infty} d\omega' \, \frac{\epsilon'(\omega')-1}{\omega'-\omega} \end{aligned}}$$

Kramers-Kronig-Relationen