

6.3 Vollständiges Differential einer Funktion in 3D

• 1D Def: $\boxed{df(x) = \frac{df}{dx} dx}$ (6.14)

$$f(x+dx) - f(x) = df + \underbrace{O(2)}_{\substack{\text{höhere Terme} \\ \text{in } dx^n \text{ mit } n \geq 2}}$$

• 3D: $f(x_1, x_2, x_3)$

Def: $\boxed{\text{vollständiges Differential}} \\ df = \frac{\partial f}{\partial x_i} dx_i$

mit $\frac{\partial f}{\partial x_i} = \frac{\partial f}{\partial x_i} \Big|_{x_j, j \neq i}$
 ↳ ... verwende bei
 Fkt. mehrerer
 Variablen
 = partielle Ableitung
 ↳ hatte $x_j \neq x_i$ fest

dann: $f(x_1+dx_1, x_2+dx_2, x_3+dx_3) - f(x_1, x_2, x_3) \\ = df + O(2)$

• Bsp: $f(r, \vartheta, \varphi) = r \sin \vartheta \cos \varphi$

$$\rightarrow df = \sin \vartheta \cos \varphi dr \\ + r \cos \vartheta \cos \varphi d\vartheta \\ - r \sin \vartheta \sin \varphi d\varphi$$

• Vektorfeld $\underline{a}(x_1, x_2, x_3)$:

$$\boxed{d\underline{a} = \frac{\partial \underline{a}}{\partial x_i} dx_i}$$
 (6.18)

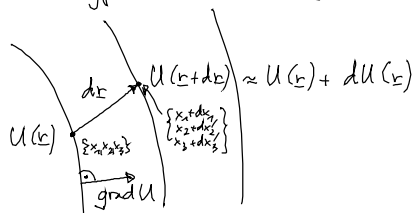
Bsp. kartesische Koordinaten

$$d\underline{a} = \frac{\partial \underline{a}}{\partial x} dx + \frac{\partial \underline{a}}{\partial y} dy + \frac{\partial \underline{a}}{\partial z} dz$$

$$\text{mit } \frac{\partial \underline{a}}{\partial x} = \frac{\partial a_x}{\partial x} \underline{e}_x + \frac{\partial a_y}{\partial x} \underline{e}_y + \frac{\partial a_z}{\partial x} \underline{e}_z \\ = \frac{\partial}{\partial x} \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}$$

6.4 Der Nabla-Operator

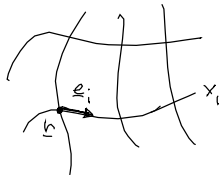
- zentrale Größe der Vektoranalysis
- führe ein über Differential eines Skalarfeldes $U(\underline{r})$:



Äquipotentialfläche
von $U(\underline{r})$: $dU=0$

$$\underline{dr} \stackrel{(6.19)}{=} \frac{\partial \underline{r}}{\partial x_i} dx_i = \left| \frac{\partial \underline{r}}{\partial x_i} \right| \underline{e}_i dx_i \quad (6.19)$$

... Wegelement,
"in finitesimaler" Differenzvektor



• einerseits:

$$dU(\underline{r}) = \frac{\partial U}{\partial x_1} dx_1 + \frac{\partial U}{\partial x_2} dx_2 + \frac{\partial U}{\partial x_3} dx_3 \quad (6.20)$$

andererseits:

Def: führe "Gradient von U " = $\text{grad} U$ als Vektor ein, so daß:

$$dU(\underline{r}) = \text{grad} U \cdot \underline{dr} \quad (6.21)$$

$$\stackrel{(6.19)}{(6.20)} \rightarrow \text{grad} U = \frac{1}{\left| \frac{\partial \underline{r}}{\partial x_i} \right|} \frac{\partial U}{\partial x_i} \underline{e}_i \quad \text{mit } \underline{e}_i \cdot \underline{e}_j = \delta_{ij} \quad (6.22)$$

... Gradientfeld von U

Beweis: $\text{grad} U \cdot \underline{dr} \stackrel{(6.22)}{(6.19)} \frac{1}{\left| \frac{\partial \underline{r}}{\partial x_i} \right|} \frac{\partial U}{\partial x_i} \underline{e}_i \cdot \left| \frac{\partial \underline{r}}{\partial x_j} \right| \underline{e}_j dx_j$

$$[\underline{e}_i \cdot \underline{e}_j = \delta_{ij}] = \frac{\partial U}{\partial x_i} dx_i$$

• (6.22) legt nahe:

Def: Nabla-Operator $\hat{=}$ Vektor-Differentialoperator

$$\nabla = \underline{e}_i \frac{1}{\left| \frac{\partial \underline{r}}{\partial x_i} \right|} \frac{\partial}{\partial x_i}, \quad \text{so daß } \text{grad} U = \nabla U \quad (6.23)$$

• entlang Äquipotentialfläche:

$$dU=0 \longrightarrow \text{grad } U \perp d\mathbf{r} \\ \longrightarrow \text{grad } U \parallel \text{Richtung maximale Änderung von } U \text{ (Skizze)}$$

• Koordinatensysteme:

a) Kartesisches Koordinatensystem: $\left| \frac{\partial \mathbf{r}}{\partial x_i} \right| = 1!$

$$d\mathbf{r} = dx \mathbf{e}_x + dy \mathbf{e}_y + dz \mathbf{e}_z \\ (6.22) \longrightarrow \boxed{\begin{aligned} \text{grad } U &= \mathbf{e}_x \frac{\partial U}{\partial x} + \mathbf{e}_y \frac{\partial U}{\partial y} + \mathbf{e}_z \frac{\partial U}{\partial z} \\ \underline{\nabla} &= \mathbf{e}_x \frac{\partial}{\partial x} + \mathbf{e}_y \frac{\partial}{\partial y} + \mathbf{e}_z \frac{\partial}{\partial z} \end{aligned}} \quad (6.24)$$

Bsp: $U \sim r^2 = x^2 + y^2 + z^2 \longrightarrow \underline{\nabla} U \sim 2x \mathbf{e}_x + 2y \mathbf{e}_y + 2z \mathbf{e}_z = 2 \mathbf{r}$

b) Zylinderkoordinaten:

$$\left. \begin{aligned} d\mathbf{r} &= \overset{(6.23)}{d\varrho} \mathbf{e}_\varrho + \varrho d\varphi \mathbf{e}_\varphi + dz \mathbf{e}_z \\ dU &= \frac{\partial U}{\partial \varrho} d\varrho + \frac{\partial U}{\partial \varphi} d\varphi + \frac{\partial U}{\partial z} dz \end{aligned} \right\} \boxed{\begin{aligned} \underline{\nabla} &= \mathbf{e}_\varrho \frac{\partial}{\partial \varrho} + \mathbf{e}_\varphi \frac{1}{\varrho} \frac{\partial}{\partial \varphi} \\ &+ \mathbf{e}_z \frac{\partial}{\partial z} \end{aligned}} \quad (6.25)$$

Bsp: $U(\varrho) \sim \ln \varrho \longrightarrow \underline{\nabla} U \sim \frac{1}{\varrho} \mathbf{e}_\varrho$ [Dimensionsanalyse: $[\underline{\nabla}] = \frac{1}{\text{Länge}} !!!$]

c) Kugelkoordinaten:

$$\boxed{\underline{\nabla} = \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\vartheta \frac{1}{r} \frac{\partial}{\partial \vartheta} + \mathbf{e}_\varphi \frac{1}{r \sin \vartheta} \frac{\partial}{\partial \varphi}} \quad (6.26)$$

Beweis: Übung

d) Rechenregeln: (Beweis: Übung)

$$(i) \underline{\nabla}(cU) = c \underline{\nabla} U, \quad c \in \mathbb{R} \\ \underline{\nabla}(U+V) = \underline{\nabla} U + \underline{\nabla} V \quad (6.27)$$

$$\underline{\nabla}(UV) = (\underline{\nabla} U)V + U(\underline{\nabla} V)$$

$$(ii) \underline{\nabla}(\mathbf{a} \cdot \mathbf{r}) = \mathbf{a}$$

$$\underline{\nabla} r = \hat{\mathbf{r}}, \quad \mathbf{r} = r \hat{\mathbf{r}} = r \mathbf{e}_r$$

$$\underline{\nabla} f(r) = \frac{df}{dr} \hat{\mathbf{r}}$$

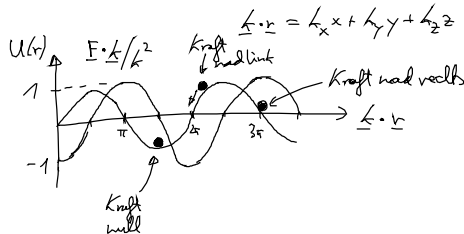
$$\text{insbes.: } \underline{\nabla} \frac{1}{r} = -\frac{\hat{\mathbf{r}}}{r^2}$$

• Richtungsableitung: „Ableitung entlang $\hat{\mathbf{v}}$ “, $|\hat{\mathbf{v}}|=1$

$$\text{Def: } \boxed{\begin{aligned} \text{Richtungsableitung: } &\hat{\mathbf{v}} \cdot \underline{\nabla} U \\ \text{so daß mit } d\mathbf{r} &= \hat{\mathbf{v}} ds: dU = (\hat{\mathbf{v}} \cdot \underline{\nabla} U) ds \end{aligned}} \quad (6.28)$$

- Beispiele: $U(\underline{r})$... potentielle Energie
 $\rightarrow \underline{F}(\underline{r}) = -\nabla U(\underline{r})$... Kraftfeld } \approx Mechanik (6.30)

(i) $U(\underline{r}) \sim \sin(\underline{k} \cdot \underline{r}) \rightarrow$ Kartes. Koordin. $\underline{F}(\underline{r}) \sim -\underline{k} \cos(\underline{k} \cdot \underline{r})$ (6.31)



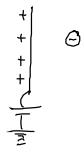
(ii) $U(\underline{r}) \sim \frac{1}{r}$... kugelsymmetr./zentral-Potential

Kugelkoordin. (6.28) $\underline{F}(\underline{r}) \sim \frac{\underline{r}}{r^2}$ (6.32)

... Kraft || radialer Richtung
 ... Gravitationskraft im Schwerfeld eines Planeten (vgl. Kap. 6.1 & 6.2)

(iii) $U(\underline{g}) \sim \ln g$... zylindersymmetr. Potential

(6.25) $\underline{F}(\underline{r}) \sim -\frac{1}{g} \underline{e}_g$ (6.33) (vgl. Kap. 6.1 & 6.2)
 Zylinderkoordin.



6.5 Divergenz

- Erinnery: $\text{grad } U(\underline{r}) = \nabla U(\underline{r})$... Vektor
 \rightarrow weitere Operationen von ∇ ?

Def:

Divergenz eines Vektorfeldes $\underline{a}(\underline{r})$
 $\text{div } \underline{a}(\underline{r}) = \nabla \cdot \underline{a}(\underline{r})$... Skalar
 ... Quellenfeld von $\underline{a}(\underline{r})$

Kartesische Koordin.

$\nabla = \underline{e}_i \frac{\partial}{\partial x_i}$, $\underline{a}(\underline{r}) = a_i(\underline{r}) \underline{e}_i$, $i = x, y, z$

$\rightarrow \nabla \cdot \underline{a} = (\underline{e}_i \frac{\partial}{\partial x_i}) \cdot (a_j(\underline{r}) \underline{e}_j)$

$[\frac{\partial}{\partial x_i} \underline{e}_j = 0] = (\frac{\partial}{\partial x_i} a_j) \underbrace{\underline{e}_i \cdot \underline{e}_j}_{\delta_{ij}}$

$\rightarrow \nabla \cdot \underline{a}(\underline{r}) = \frac{\partial}{\partial x} a_x + \frac{\partial}{\partial y} a_y + \frac{\partial}{\partial z} a_z$

Bsp. 1: $\underline{a}(\underline{r}) = \underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ (6.36)

$$\rightarrow \nabla \cdot \underline{r} = 1 + 1 + 1 = 3$$

Bsp 2: Kugel-symmetr. Quellen-/Senkenfeld

$$\underline{a}(\underline{r}) = a(r) \underline{e}_r = \frac{a(r)}{r} \underline{r} \quad \underline{e}_r = \frac{\underline{r}}{r}$$

$$\nabla \cdot \underline{a}(\underline{r}) = \frac{a(r)}{r} \underbrace{\nabla \cdot \underline{r}}_3 + \frac{\partial a}{\partial r} \underbrace{\nabla \cdot (\underline{r}/r)}_{\frac{\partial}{\partial x_i}(r) x_i = \hat{r} \cdot \underline{r} = r}$$

$$= 2 \frac{a(r)}{r} + \frac{\partial a}{\partial r} \quad (6.38)$$

$$\text{für } a(r) = \frac{1}{r^2} \quad [\text{s. Kap. 6.2}]$$

$$\nabla \cdot \underline{a}(\underline{r}) = 0! \quad (6.39) \quad \text{für } r \neq 0 \quad (\text{Singularität})$$

also: Punktmasse/-ladung bei $r=0$
erzeugt Feld, sonst keine Quelle!