6.3 Vollständiges Differential einer Funktion in 3D

- $\boxed{1D}$ Def: $\boxed{df(x) = \frac{df}{dx}\, dx}$ (6.14)

$$f(x+dx) - f(x) = df + \underbrace{O(2)}_{\text{höhere Terme in } dx^n \text{ mit } n \geq 2}$$

- $\boxed{3D}$: $f(x_1, x_2, x_3)$

Def: $\boxed{\text{vollständiges Differential } df = \frac{\partial f}{\partial x_i}\, dx_i}$

mit $\frac{\partial f}{\partial x_i} = \left.\frac{\partial f}{\partial x_i}\right|_{x_j,\, j \neq i}$

∂... verwende bei Fkt. mehrerer Veränderlicher = partielle Ableitung

halte $x_j \neq x_i$ fest

dann: $f(x_1 + dx_1, x_2 + dx_2, x_3 + dx_3) - f(x_1, x_2, x_3)$

$$= df + O(2)$$

- Bsp: $f(r, \vartheta, \varphi) = r \sin\vartheta \cos\varphi$

$$\rightarrow df = \sin\vartheta \cos\varphi\, dr + r\cos\vartheta \cos\varphi\, d\vartheta - r \sin\vartheta \sin\varphi\, d\varphi$$

- Vektorfeld $\underline{a}(x_1, x_2, x_3)$:

$\boxed{d\underline{a} = \frac{\partial \underline{a}}{\partial x_i}\, dx_i}$ (6.18)

Bsp. kartesische Koordinaten

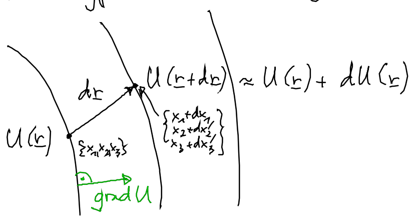
$$d\underline{a} = \frac{\partial \underline{a}}{\partial x} dx + \frac{\partial \underline{a}}{\partial y} dy + \frac{\partial \underline{a}}{\partial z} dz$$

mit $\frac{\partial \underline{a}}{\partial x} = \frac{\partial a_x}{\partial x} \underline{e}_x + \frac{\partial a_y}{\partial x} \underline{e}_y + \frac{\partial a_z}{\partial x} \underline{e}_z$

$$= \frac{\partial}{\partial x} \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}$$

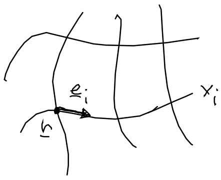
6.4 Der Nabla-Operator

- zentrale Größe der Vektoranalysis
- Führe ein über Differential eines Skalarfeldes $U(\underline{r})$:



Äquipotentialflächen von $U(\underline{r})$: $dU = 0$

$$\boxed{d\underline{r} \overset{}{\underset{(6.18)}{=}} \frac{\partial \underline{r}}{\partial x_i}\, dx_i = \left|\frac{\partial \underline{r}}{\partial x_i}\right| \underline{e}_i\, dx_i} \quad (6.19)$$



... Wegelement,
„infinitesimaler" Differenzvektor

- einerseits:

$$dU(\underline{r}) = \frac{\partial U}{\partial x_1} dx_1 + \frac{\partial U}{\partial x_2} dx_2 + \frac{\partial U}{\partial x_3} dx_3 \quad (6.20)$$

andererseits:

Def: Führe „Gradient von U" = grad U als Vektor ein, so daß:

$$dU(\underline{r}) = \text{grad}\, U \cdot d\underline{r} \quad (6.21)$$

$$\xrightarrow[(6.20)]{(6.19)} \quad \boxed{\text{grad}\, U = \frac{1}{\left|\frac{\partial \underline{r}}{\partial x_i}\right|} \frac{\partial U}{\partial x_i}\, \underline{e}_i \quad \text{mit } \underline{e}_i \cdot \underline{e}_j = \delta_{ij}} \quad (6.22)$$

... Gradientfeld von U

Beweis: $$\text{grad}\, U \cdot d\underline{r} \underset{(6.19)}{\overset{(6.22)}{=}} \frac{1}{\left|\frac{\partial \underline{r}}{\partial x_i}\right|} \frac{\partial U}{\partial x_i}\, \underline{e}_i \cdot \left|\frac{\partial \underline{r}}{\partial x_j}\right| \underline{e}_j\, dx_j$$

$$[\underline{e}_i \cdot \underline{e}_j = \delta_{ij}] = \frac{\partial U}{\partial x_i}\, dx_i$$

- (6.22) legt nahe:

Def: Nabla-Operator ≙ Vektor-Differentialoperator

$$\underline{\nabla} = \underline{e}_i \frac{1}{\left|\frac{\partial \underline{r}}{\partial x_i}\right|} \frac{\partial}{\partial x_i}, \quad \text{so daß } \text{grad}\, U = \underline{\nabla} U \quad (6.23)$$

- entlang Äquipotentialfläche:

$$dU = 0 \longrightarrow \mathrm{grad}\, U \perp d\underline{r}$$

$$\longrightarrow \mathrm{grad}\, U \parallel \text{Richtung maximaler Änderung von } U \text{ (Skizze)}$$

- Koordinatensysteme:

a) Kartesische Koordinatensystem: $\left|\frac{\partial \underline{r}}{\partial x_i}\right| = 1!$

$$d\underline{r} = dx\, \underline{e}_x + dy\, \underline{e}_y + dz\, \underline{e}_z$$

$$(6.22) \longrightarrow \boxed{\begin{aligned} \mathrm{grad}\, U &= \underline{e}_x \frac{\partial U}{\partial x} + \underline{e}_y \frac{\partial U}{\partial y} + \underline{e}_z \frac{\partial U}{\partial z} \\ \underline{\nabla} &= \underline{e}_x \frac{\partial}{\partial x} + \underline{e}_y \frac{\partial}{\partial y} + \underline{e}_z \frac{\partial}{\partial z} \end{aligned}} \quad (6.24)$$

Bsp: $U \sim r^2 = x^2 + y^2 + z^2 \longrightarrow \underline{\nabla} U \sim 2x\, \underline{e}_x + 2y\, \underline{e}_y + 2z\, \underline{e}_z$
$= 2\, \underline{r}$

b) Zylinderkoordinaten:

$$\left.\begin{aligned} d\underline{r} &\overset{(5.9)}{=} d\varrho\, \underline{e}_\varrho + \varrho\, d\varphi\, \underline{e}_\varphi + dz\, \underline{e}_z \\ dU &= \frac{\partial U}{\partial \varrho} d\varrho + \frac{\partial U}{\partial \varphi} d\varphi + \frac{\partial U}{\partial z} dz \end{aligned}\right\} \quad \boxed{\begin{aligned} \underline{\nabla} &\overset{(6.23)}{=} \underline{e}_\varrho \frac{\partial}{\partial \varrho} + \underline{e}_\varphi \frac{1}{\varrho}\frac{\partial}{\partial \varphi} \\ &\quad + \underline{e}_z \frac{\partial}{\partial z} \end{aligned}} \quad (6.25)$$

Bsp: $U(\varrho) \sim \ln \varrho \longrightarrow \underline{\nabla} U \sim \frac{1}{\varrho}\, \underline{e}_\varrho$ [Dimensionsanalyse: $[\underline{\nabla}] = \frac{1}{\text{Länge}}$!!!]

c) Kugelkoordinaten:

$$\boxed{\underline{\nabla} = \underline{e}_r \frac{\partial}{\partial r} + \underline{e}_\vartheta \frac{1}{r}\frac{\partial}{\partial \vartheta} + \underline{e}_\varphi \frac{1}{r \sin\vartheta}\frac{\partial}{\partial \varphi}} \quad (6.26)$$

Beweis: Übungen

d) Rechenregeln: (Beweis: Übungen)

(i) $\underline{\nabla}(cU) = c\, \underline{\nabla} U, \quad c \in \mathbb{R}$ (6.27)

$\underline{\nabla}(U + V) = \underline{\nabla} U + \underline{\nabla} V$

$\underline{\nabla}(UV) = (\underline{\nabla} U) V + U (\underline{\nabla} V)$

(ii) $\underline{\nabla}(\underline{a} \cdot \underline{r}) = \underline{a}$

$\underline{\nabla} r = \hat{\underline{r}}, \quad \underline{r} = r\hat{\underline{r}} = r\, \underline{e}_r$

$\underline{\nabla} f(r) = \frac{\partial f}{\partial r}\, \hat{\underline{r}}$

insbes.: $\underline{\nabla} \frac{1}{r} = -\frac{\hat{\underline{r}}}{r^2}$

- Richtungsableitung: „Ableitung entlang $\hat{\underline{\nu}}$", $|\hat{\underline{\nu}}| = 1$

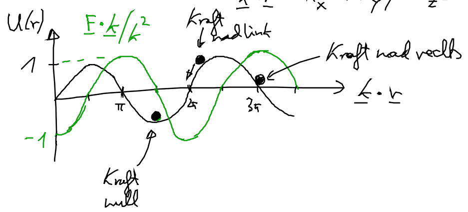
Def: $\boxed{\begin{array}{l} \text{Richtungsableitung: } \hat{\underline{\nu}} \cdot \underline{\nabla} U \\ \text{so daß mit } d\underline{r} = \hat{\underline{\nu}}\, ds: \quad dU = (\hat{\underline{\nu}} \cdot \nabla U)\, ds \end{array}}$ (6.29)

• Beispiele: $\underline{U}(\underline{r})$ ... potentielle Energie

$\longrightarrow \underline{F}(\underline{r}) = -\underline{\nabla} U(\underline{r})$ ... Kraftfeld } s. Mechanik (6.30)

(i) $U(\underline{r}) \sim \sin(\underline{k}\cdot\underline{r}) \xrightarrow[\text{Koord.}]{\text{kartes.}} \underline{F}(\underline{r}) \sim -\underline{k}\cos(\underline{k}\cdot\underline{r})$ (6.31)

$\underline{k}\cdot\underline{r} = k_x x + k_y y + k_z z$



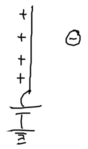
(ii) $U(r) \sim \frac{1}{r}$ ... kugelsymmetr. / Zentral-Potential

$\xrightarrow[(6.28)]{\text{Kugelkoord.}} \underline{F}(\underline{r}) \sim \frac{\underline{e}_r}{r^2}$ (6.32)

... Kraft ∥ radiale Richtg.
... Gravitationskraft im Schwerefeld eines Planeten (vgl. Kap. 6.1 & 6.2)

(iii) $U(\varrho) \sim \ln\varrho$ ... zylindersymmetr. Potential

$\xrightarrow[\text{Zylinderkoord.}]{(6.25)} \underline{F}(\underline{r}) \sim -\frac{1}{\varrho}\underline{e}_\varrho$ (6.33) (vgl. Kap. 6.1 & 6.2)



6.5 Divergenz

• Erinnerg.: $\text{grad}\, U(\underline{r}) = \underline{\nabla} U(\underline{r})$ ... Vektor

$\longrightarrow$ weitere Operationen von $\underline{\nabla}$?

• Def.: Divergenz eines Vektorfeldes $\underline{a}(\underline{r})$

$$\boxed{\text{div}\, \underline{a}(\underline{r}) = \underline{\nabla}\cdot\underline{a}(\underline{r}) \quad \text{... Skalar}}$$

... Quellenfeld von $\underline{a}(\underline{r})$

• kartesische Koord.:

$\underline{\nabla} = \underline{e}_i \frac{\partial}{\partial x_i}$, $\underline{a}(\underline{r}) = a_i(\underline{r})\,\underline{e}_i$, $i = x, y, z$

$\longrightarrow \underline{\nabla}\cdot\underline{a} = \left(\underline{e}_i \frac{\partial}{\partial x_i}\right)\cdot\left(a_j(\underline{r})\,\underline{e}_j\right)$

$\left[\frac{\partial}{\partial x_i}\underline{e}_j = 0\right] = \left(\frac{\partial}{\partial x_i} a_j\right) \underbrace{\underline{e}_i\cdot\underline{e}_j}_{\delta_{ij}}$

$$\longrightarrow \boxed{\underline{\nabla}\cdot\underline{a}(\underline{r}) = \frac{\partial}{\partial x} a_x + \frac{\partial}{\partial y} a_y + \frac{\partial}{\partial z} a_z}$$

• Bsp. 1: $\underline{a}(\underline{r}) = \underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ (6.36)

$\longrightarrow \quad \underline{\nabla} \cdot \underline{r} = 1 + 1 + 1 = 3$

Bsp 2: Kugelsymmetr. Quellen-/Senkenfeld

$$\underline{a}(\underline{r}) = a(r)\, \underline{e}_r = \frac{a(r)}{r}\, \underline{r} \qquad \underline{e}_r = \frac{\underline{r}}{r}$$

$$\underline{\nabla} \cdot \underline{a}(\underline{r}) = \frac{a(r)}{r} \underbrace{\underline{\nabla} \cdot \underline{r}}_{3} + \frac{\partial\, a/r}{\partial r} \underbrace{\underline{\nabla} \cdot (\hat{r}\, \underline{r})}_{\frac{\partial}{\partial x_i}(r)\, x_i = \hat{\underline{r}} \cdot \underline{r} = r}$$

$$= 2\,\frac{a(r)}{r} + \frac{\partial a}{\partial r} \qquad (6.38)$$

für $a(r) = \frac{1}{r^2}$ [s. Kap. 6.2]

$$\underline{\nabla} \cdot \underline{a}(\underline{r}) = 0! \quad (6.39) \qquad \text{für } r \neq 0 \text{ (Singularität)}$$

also: Punktmasse/-ladung bei $r = 0$ erzeugt Feld, sonst keine Quelle!