7.4 Volumenintegrale

Def.:
$$\boxed{\int f(\underline{r})\,dV \xleftarrow{\Delta V_i \to dV} \sum_{\text{"}i\in V\text{"}} f(\underline{r}_i)\,\Delta V_i} \quad (7.24)$$

- Kartesische Koordinaten:

$$\boxed{dV = dx\,dy\,dz}$$

- beliebige Koord.: x_1, x_2, x_3

(1) Koord.trafo: $\underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x\,(x_1,x_2,x_3) \\ y\,(\;\;"\;\;) \\ z\,(\;\;"\;\;) \end{pmatrix} \quad (7.26)$

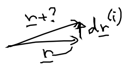
(2) Welches Volumen gehört zu: $dx_1\,dx_2\,dx_3$

Bsp: $dr\,d\vartheta\,d\varphi$ ... Einheit Länge!
nicht Volumen
$\longrightarrow$ Vorfaktor?

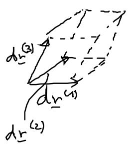
Verschiebungsvektor für dx_i:

$$d\underline{r}^{(i)} = \underbrace{\frac{\partial \underline{r}}{\partial x_i}}_{\sim \underline{e}_i} dx_i = \frac{\partial(x,y,z)}{\partial x_i}\,dx_i \quad (7.27)$$

$\underline{r}$ ... kartesisch



$\longrightarrow$ Spatprodukt



$$dV = d\underline{r}^{(1)} \cdot (d\underline{r}^{(2)} \times d\underline{r}^{(3)})$$
$$\overset{(7.27)}{=} \frac{\partial \underline{r}}{\partial x_1} \cdot \left(\frac{\partial \underline{r}}{\partial x_2} \times \frac{\partial \underline{r}}{\partial x_3}\right) dx_1\,dx_2\,dx_3 \quad (7.28)$$

(3) Führe ein:
Jacobi-Matrix: $\boxed{\underline{\underline{F}} = \frac{\partial(x,y,z)}{\partial(x_1,x_2,x_3)} = \left(\frac{\partial \underline{r}}{\partial x_1}, \frac{\partial \underline{r}}{\partial x_2}, \frac{\partial \underline{r}}{\partial x_3}\right)} \quad (7.29)$

Spaltenvektoren

$\xrightarrow[\text{mit (7.29)}]{(7.28)}$
& Spatprodukt
$= \det \underline{\underline{F}}$

$$\boxed{\begin{aligned} dV &= \left|\frac{\partial(x,y,z)}{\partial(x_1,x_2,x_3)}\right| dx_1\,dx_2\,dx_3 \\ &= \underbrace{\det \underline{\underline{F}}}_{\text{Funktionaldeterminante}}\; dx_1\,dx_2\,dx_3 \end{aligned}} \quad (7.30)$$

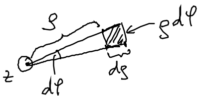
Bsp: (1) Zylinderkoord.: $x_i = \rho, \varphi, z$

$$\left.\begin{aligned} x &= \rho\cos\varphi \\ y &= \rho\sin\varphi \\ z &= z \end{aligned}\right\} \longrightarrow \underline{\underline{F}} = \begin{pmatrix} \cos\varphi & -\rho\sin\varphi & 0 \\ \sin\varphi & \rho\cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (7.31)$$

$$\longrightarrow \det \underline{\underline{F}} = 1\,\rho\,(\cos^2\varphi - (-)\sin^2\varphi)$$

$$= \varrho \quad (7.32)$$

$$\longrightarrow \boxed{dV = \varrho\, d\varrho\, d\varphi\, dz} \quad (7.33)$$

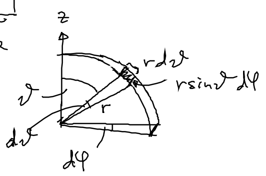


(2) Kugelkoord.: $x_i = r, \vartheta, \varphi$

$$\longrightarrow \det \underline{\underline{F}} \overset{\text{o.B.}}{=} r^2 \sin\vartheta \quad (7.34)$$

$$\longrightarrow \boxed{dV = \underbrace{r^2 \sin\vartheta\, d\vartheta\, d\varphi}_{\text{Grundfläche auf Kugelschale}} \times \underbrace{dr}_{\text{Höhe}}} \quad (7.35)$$

[vgl. (7.19)]



- Bsp: Volumen V_k einer Kugel mit Radius R

$\longrightarrow$ Kugelkoordi. mit $f(\underline{r}) = 1$ in (7.24)

$$V_k = \int_{V_k} dV = \underbrace{\int_0^R \int_0^{2\pi} \int_0^{\pi} r^2 \sin\vartheta\, d\vartheta\, d\varphi\, dr}_{\text{3-fach Integral}}$$

$$= \underbrace{\left(\int_0^R r^2 dr\right)}_{\left.\frac{r^3}{3}\right|_0^R = \frac{R^3}{3}} \underbrace{\left(\int_0^{2\pi} d\varphi\right)}_{2\pi} \underbrace{\left(\int_0^{\pi} \sin\vartheta\, d\vartheta\right)}_{\int_{-1}^{1} d\cos\vartheta = 2}$$

$$= \frac{4\pi}{3} R^3 \,!$$

7.5 Gaußscher Satz

- Satz: Für Quellen von $\underline{a}$ in V gilt:

$$\boxed{\int_V \operatorname{div} \underline{a}\, dV = \int_{\partial V} \underline{a} \cdot d\underline{f} \quad (7.36)}$$

... Fluß durch Oberfläche ∂V

wichtig: (1) $\operatorname{div} \underline{a}$ definiert in ganz V
(2) $d\underline{f}$ zeigt aus V heraus

• Beweis:

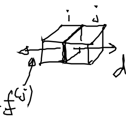
1. Nähere V durch „viele" Quader ΔV_i 

„keine Zeichnung"

2. $\int_V \operatorname{div} \underline{a}\, dV = \sum_i \operatorname{div} \underline{a}(\underline{r}_i)\, \Delta V_i$ (über alle Quader)

3. $\operatorname{div} \underline{a}(\underline{r}_i)\, \Delta V_i =$ "$\oint_{\partial(\Delta V_i)} \underline{a} \cdot d\underline{f}^{(i)}$" [vgl. Kap. 6.5, Gl. (6.43)]

4. benachbarte Vol. elemente:



$$d\underline{f}^{(i)} = - d\underline{f}^{(j)}$$

$$\longrightarrow \underline{a} \cdot \underbrace{d\underline{f}^{(i)}}_{-d\underline{f}^{(j)}} + \underline{a} \cdot d\underline{f}^{(j)} = 0$$

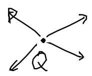
also: in $\sum_i \operatorname{div} \underline{a}(\underline{r}_i)\, \Delta V_i = \sum_i$ "$\int_{\partial(\Delta V_i)} \underline{a} \cdot d\underline{f}^{(i)}$"

nur „freiliegende" Oberflächen der ΔV_i tragen bei

$\longrightarrow$ Oberfläche ∂V von V

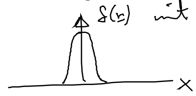
$$\longrightarrow \sum_i \text{"}\oint_{\partial(\Delta V_i)} \underline{a} \cdot d\underline{f}^{(i)}\text{"} = \int_{\partial V} \underline{a} \cdot d\underline{f} \quad \text{qed}$$

• Anwendung: E-Feld einer Pkt. Ladung Q



Maxwell: $\operatorname{div} \underline{E} \sim \underbrace{Q\,\delta(\underline{r})}_{\text{Ladungsdichte}}$

$\delta(\underline{r})$ mit $\int \delta(\underline{r})\, dV = 1$



(i) $\boxed{\underline{E} = E(r)\, \underline{e}_r\ !}$ (7.38)

(ii) $\int_{V_K} \operatorname{div} \underline{E}\, dV \sim \int_{V_K} Q\, \delta(\underline{r}) = Q$

Kugel um $\underline{r} = 0$

(iii) $\int_{\partial V_K} \underline{E} \cdot d\underline{f} \overset{(7.38)}{=} \int_{\partial V_K} E(r)\, \underline{e}_r \cdot \underline{e}_r\, df$

$$= \int_{\partial V_K} E(r)\, df = E(r) \int_{\partial V_K} df = E(r)\, 4\pi r^2$$

Gauß: (ii) = (iii) $\longrightarrow$ $\boxed{E(r) \sim \frac{Q}{r^2}}$! (7.39)