Wdh

$|\dot\psi\rangle = -\frac{i}{\hbar} H |\psi\rangle \rightsquigarrow |\psi(t)\rangle = U(t)|\psi_0\rangle$

$\rightsquigarrow \dot U(t) = -\frac{i}{\hbar} H U(t) \rightsquigarrow \dot U^\dagger(t) = +\frac{i}{\hbar} U^\dagger(t) H(t)$

$$\rho = |\psi(t)\rangle\langle\psi(t)| = U(t)|\psi_0\rangle\langle\psi_0|U^\dagger(t)$$

$$\dot\rho = -\frac{i}{\hbar} H(t)\,\rho(t) + \rho\left(+\frac{i}{\hbar}H(t)\right) \longrightarrow \frac{i}{\hbar}[H(t), \rho]$$

- DM $\{|\Phi_i\rangle\}$ $\quad \langle\Phi_i|\Phi_i\rangle = 1$

$0 \le P_i \le 1 \qquad \sum_i P_i = 1 \qquad \rho = \sum_i P_i |\Phi_i\rangle\langle\Phi_i|$

jede DM erfüllt $\mathrm{Tr}\{\rho\} = 1$

$\rho = \rho^\dagger$

EW von ρ

ρ ist positiv semidefinit $\langle\psi|\rho|\psi\rangle \ge 0 \qquad \rho_i \ge 0$

- $\mathrm{Tr}\{\rho\} = \sum_i P_i \mathrm{Tr}\{|\Phi_i\rangle\langle\Phi_i|\} = \sum_i P_i = 1$
- $\langle\psi|\rho|\psi\rangle = \sum_i P_i \langle\psi|\Phi_i\rangle\langle\Phi_i|\psi\rangle = \sum_i P_i \underbrace{|\langle\psi|\Phi_i\rangle|^2}_{\ge 0} \ge 0$
- Quanten-Verallg. für diskrete WS-Vert. $\{P_i\}$
- ERW $\boxed{\langle A\rangle = \mathrm{Tr}\{A\rho\} = \mathrm{Tr}\{\rho A\}} = \sum_i P_i \langle\Phi_i|A|\Phi_i\rangle$

- "reine" Zustände $\rho^2 = \rho \qquad \rho = |\psi\rangle\langle\psi| \qquad$ z.B. $\rho = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

"gemischte" " : Gegenteil $\rho^2 \ne \rho$

$|0\rangle \hat= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\rho = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \hat= |\psi\rangle\langle\psi|$

$[|\psi\rangle = \frac{1}{\sqrt2}(|0\rangle + |1\rangle)]$

- Messprozess: $\{M_m\}$: $\sum_m M_m^\dagger M_m = \mathbb{1}$

projektive Messung $M_m \to |m\rangle\langle m|$

$$\rho \xrightarrow{(m)} \frac{M_m \rho M_m^\dagger}{P_m} \qquad P_m = \mathrm{Tr}\{M_m \rho M_m^\dagger\} = \mathrm{Tr}\{M_m^\dagger M_m \rho\}$$

- "Populationen": Diagonalelemente von ρ } basis-abhängig
"Kohärenzen": Off-Diagonalelemente

3.1.3 Allgemeinste Entwicklung

$$\boxed{\text{Kraus-Abb.:} \quad \rho' = \sum_\tau K_\tau \rho K_\tau^\dagger \qquad \sum_\tau K_\tau^\dagger K_\tau = \mathbb{1}}$$

- $\rho'^\dagger = \rho'$ • $\mathrm{Tr}\{\rho'\} = \sum_\tau \mathrm{Tr}\{K_\tau^\dagger K_\tau \rho\} = \mathrm{Tr}\{\rho\}$

- $\langle\psi|\rho'|\psi\rangle = \sum_\tau \langle\psi|K_\tau \rho K_\tau^\dagger|\psi\rangle$
$= \sum_\tau\sum_\lambda |\langle\psi|K_\alpha|v_\lambda\rangle|^2 \cdot \rho_\lambda \geq 0$

$$\rho = \sum_\lambda \underbrace{\rho_\lambda}_{\geq 0} |v_\lambda\rangle\langle v_\lambda| \qquad \rho_\lambda: \text{EW von } \rho$$

- Lindblad-ME: $\dot\rho = -i[H,\rho] + \sum_\alpha \left[L_\alpha \rho L_\alpha^\dagger - \frac{1}{2} L_\alpha^\dagger L_\alpha \rho - \frac{1}{2}\rho L_\alpha^\dagger L_\alpha\right]$

3.1.4 Entropie

$$\boxed{\begin{array}{l}\text{von-Neumann-Entropie} \\ S_{vN}(\rho) = -\mathrm{Tr}\{\rho \ln \rho\}\end{array}}$$

$\rho = \sum_\alpha \rho_\alpha |v_\alpha\rangle\langle v_\alpha|$ (ρ_α: EW, $|v_\alpha\rangle$: EV)
$\ln\rho = \sum_\alpha \ln(\rho_\alpha) |v_\alpha\rangle\langle v_\alpha|$
$S_{vN}(\rho) = \sum_\alpha -\rho_\alpha \ln \rho_\alpha = S_{Sh}(\{\rho_\alpha\})$

$\rho \longrightarrow \rho_D = \sum_i |i\rangle\langle i|\rho|i\rangle\langle i| = \sum_i \langle i|\rho|i\rangle\, |i\rangle\langle i|$ diag. Anteil der DM

z.B. $\rho = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} \longrightarrow \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

Wie verhält sich $S_{vN}(\rho)$ zu $S_{vN}(\rho_D)$

- $D(\rho\|\sigma) = \mathrm{Tr}\{\rho\ln\rho - \rho\ln\sigma\} \underset{\text{Klein}}{\geq} 0$

ρ, σ: zwei gültige DM

- klar: $D(\rho\|\rho) = 0$
$D(\rho\|\sigma) \neq D(\sigma\|\rho)$

$D(\rho\|\rho_D) = -S_{vN}(\rho) - \mathrm{Tr}\{\rho\ln\rho_D\} = -S_{vN}(\rho) - \mathrm{Tr}\{\rho[\sum_i \ln \underbrace{\rho_{ii}}_{\text{Diagonalelemente}} |i\rangle\langle i|]\}$
$= -S_{vN}(\rho) + S_{vN}(\rho_D) \geq 0$

$\rightsquigarrow \boxed{S_{vN}(\rho_D) \geq S_{vN}(\rho)}$

$S\left(\frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\right) = 0$

$S\left(\frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right) = \ln 2$

3.1.5 Gibbs-Zustand

a.) Maximiere S unter NB: $\mathrm{Tr}\{\rho\} = 1$

$\rho = \sum_n \underbrace{\rho_n}_{\text{EW}} \underbrace{|n\rangle\langle n|}_{\text{EV}} \rightsquigarrow \rho_n = \frac{1}{N} \qquad \rho^* = \sum_n \frac{1}{N} |n\rangle\langle n| = \frac{1}{N}\cdot\mathbb{1}$

$U U U^+ = \mathbb{1}$ ⇝ gilt basis-unabhängig

b.) Fordere $Tr\{\rho\} = 1$, $Tr\{\rho \cdot H\} = u$

$$\tilde{S} = S_{vN}(\rho) + \lambda\left[Tr\{\rho\} - 1\right] + \beta\left[Tr\{H\rho\} - u\right]$$ ⇝ extremiere

$$H = \sum_\alpha E_\alpha |\alpha\rangle\langle\alpha| \qquad \langle\alpha|\rho|\alpha\rangle = \rho_{\alpha\alpha}$$

EW von H EV von H

$$\tilde{S} = S_{vN}(\rho) + \lambda\left[\sum_\alpha \rho_{\alpha\alpha} - 1\right] + \beta\left[\sum_\alpha \rho_{\alpha\alpha} \cdot E_\alpha - u\right]$$

=> NB hängen nur von ρ_D in Energie-EB ab

⇝ $\alpha \neq \alpha'$: $\langle\alpha|\rho|\alpha'\rangle = 0$

⇝ ρ^* ist diagonal in Energie-EB $\qquad \rho_{\alpha\alpha} = \frac{e^{-\beta E_\alpha}}{Z_c}$

$$\boxed{\rho_c = \frac{e^{-\beta \hat{H}}}{Tr\{e^{-\beta \hat{H}}\}}}$$

kanonischer Gibbs-Zustand

$$\langle A \rangle_c = Tr\{\hat{A}\, \rho_c\}$$

c.) $Tr\{\rho\} \stackrel{!}{=} 1 \qquad Tr\{\hat{H}\rho\} = u \qquad Tr\{\hat{N}\rho\} = N \qquad \hat{N} = a^+ a$

falls $[\hat{H}, \hat{N}] = 0$

$$\boxed{\rho_{gc} = \frac{e^{-\beta(\hat{H} - \mu \hat{N})}}{Tr\{e^{-\beta(\hat{H} - \mu\hat{N})}\}}}$$

großk. Gibbs-Zustand

• Diskussion:

+ $u = -\partial_\beta \ln Z_c = \frac{Tr\{H \cdot e^{-\beta H}\}}{Tr\{e^{-\beta H}\}}$

+ falls $H = H_0 + b \cdot H_1$ $\qquad$ allg: $[H_0, H_1] \neq 0$

$$\langle H_1 \rangle_c = \frac{-1}{\beta} \cdot \partial_b \ln Z_c = -\frac{1}{\beta}\frac{1}{Z_c} Tr\{\partial_b e^{-\beta(H_0 + b \cdot H_1)}\}$$

$$= -\frac{1}{\beta}\frac{1}{Z_c}\sum_{n=0}^{\infty} Tr\left\{\partial_b \frac{(-\beta)^n}{n!}(H_0 + b \cdot H_1)^n\right\} = -\frac{1}{\beta} \cdot \frac{1}{Z_c}\sum_{n=0}^{\infty}\sum_{k=1}^{n} Tr\left\{\frac{(-\beta)^n}{n!}(H_0 + b \cdot H_1)^k H_1 \times\right.$$

$$\left. \times (H_0 + b H_1)^{n-k-1}\right\}$$

$$= Tr\{H_1 \rho_c\} =$$

- Trick geht nur für das 1. Moment von H_1

3.1.6. HO

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 = \hbar\omega\left(a^\dagger a + \tfrac{1}{2}\right)$$

$[a, a^\dagger] = 1$

$[\hat{x}, \hat{p}] = i\hbar$

$a^\dagger a|n\rangle = n|n\rangle$

$$\boxed{\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\,(a^\dagger + a) \qquad \hat{p} = i\sqrt{\frac{m\hbar\omega}{2}}\,(a^\dagger - a)}$$

$$Z_c = \mathrm{Tr}\{e^{-\beta\hat{H}}\} = \sum_{n=0}^{\infty}\langle n|e^{-\beta\hbar\omega(a^\dagger a+1/2)}|n\rangle$$
$$= \sum_{n=0}^{\infty} e^{-\beta\hbar\omega(n+1/2)} = \frac{1}{e^{+\beta\hbar\omega/2} - e^{-\beta\hbar\omega/2}}$$

$$\boxed{\rho_c = \frac{e^{-\beta\hbar\omega(a^\dagger a+1/2)}}{Z_c} = \sum_{n=0}^{\infty}\underbrace{\frac{e^{-\beta\hbar\omega(n+1/2)}}{Z_c}}_{P_n}|n\rangle\langle n|}$$

$\langle x\rangle \propto \mathrm{Tr}\{(a+a^\dagger)\rho_c\}$

$a|n\rangle = \sqrt{n}\,|n-1\rangle$

$a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$

$a|n\rangle\langle n| = \sqrt{n}\,|n-1\rangle\langle n|$

$\mathrm{Tr}\{P_n|n-1\rangle\langle n|\sqrt{n} + P_n|n+1\rangle\langle n|\sqrt{n+1}\}$

$\langle x\rangle = 0 = \langle p\rangle$

$\mathrm{Tr}\{|n-1\rangle\langle n|\} = \sum_m \langle m|n-1\rangle\underbrace{\langle n|m\rangle}_{\delta_{nm}} = \underbrace{\langle n|n-1\rangle}_{=0}$

$$\langle x^2\rangle = \frac{\hbar}{2m\omega}\mathrm{Tr}\{(a+a^\dagger)^2\rho_c\}$$
$$= \frac{\hbar}{2m\omega}\mathrm{Tr}\{(\cancel{a^2} + \cancel{(a^\dagger)^2} + aa^\dagger + a^\dagger a)\rho_c\}$$
$$= \frac{\hbar}{2m\omega}\sum_{n=0}^{\infty}\langle n|(a^\dagger a + a^\dagger a + 1)\rho_c\}$$
$$= \frac{\hbar}{2m\omega}\sum_{n=0}^{\infty}(2n+1)\cdot\frac{e^{-\beta\hbar\omega(n+1/2)}}{Z_c} = \frac{\hbar}{2m\omega}\coth\left(\frac{\beta\hbar\omega}{2}\right)$$

$\xrightarrow{\text{große } T} \frac{k_B T}{m\omega^2}$

$\xrightarrow{\text{tiefe } T} \frac{\hbar}{2m\omega}$

$\left\langle \frac{1}{2}m\omega^2\hat{x}^2\right\rangle = \frac{k_B T}{2}$

$\left\langle \frac{1}{2}m\omega^2\hat{x}^2\right\rangle = \frac{\hbar\omega}{4}$

$\hat{=}$ Hälfte der GZ-Energie

auch bei $T \to 0$ ist $\langle x^2\rangle > 0$

anders $H = \frac{\hat{p}^2}{2m} + \underbrace{\left(\frac{1}{2}m\omega^2\right)}_{b}\hat{x}^2$

$$\langle x^2\rangle = -\frac{1}{\beta}\frac{\partial}{\partial\left(\frac{1}{2}m\omega^2\right)}\ln Z_c = \frac{-1}{\beta m\omega^2}\partial_\omega \ln Z_c$$
$$= -\frac{1}{\beta}\frac{\partial}{\frac{1}{2}m\,\partial\omega^2}$$