Wdh

- Planck'sches Strahlungsgesetz

$$\frac{du}{d\omega} = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{e^{\beta\hbar\omega}-1}$$

Ableitung: $\varepsilon(k) = \hbar c \cdot k$ $\hat{=}$ ultrarel. Quantengas

$p_{\text{ultrarel.}} = \frac{2}{3} \cdot \frac{u}{V}$ $\qquad p_{\text{rel}} = \frac{1}{3} \cdot \frac{u}{V}$

$$\frac{u}{V} = 2 \cdot \frac{1}{2\pi^2} \int_0^\infty \frac{\hbar c\, k^3 dk}{e^{\beta\hbar c k}-1} \qquad \frac{N}{V} = 2 \cdot \frac{1}{2\pi^2} \int \frac{k^2 dk}{e^{\beta\hbar c k}-1}$$

2 Polarisationen $\qquad \mu_{\text{phot}} = 0$

- Quanten-Ising-Modell

$\alpha \in \{x, y, z\}$

$$\sigma_i^\alpha = \mathbb{1}_1 \otimes \ldots \otimes \mathbb{1}_{i-1} \otimes \sigma^\alpha \otimes \mathbb{1}_{i+1} \otimes \ldots \otimes \mathbb{1}_N$$

$$H_{\text{Ising}} = \underbrace{-g \sum_{i=1}^N \sigma_i^x}_{H_0} \underbrace{- J \sum_{i=1}^N \sigma_i^z \sigma_{i+1}^z}_{H_1} \qquad \sigma_{N+1}^z = \sigma_N^z$$

$$[H_0, H_1] \neq 0$$

V, W seien Hilberträume

$\{|v_i\rangle\}$ $\quad \{|w_i\rangle\}$

$V \otimes W$ ist auch ein HR

$\{|v_i\rangle \otimes |w_j\rangle\}$ ist eine Basis in $V \otimes W$

$$C = \sum_\tau A_\alpha \otimes B_\tau$$

B_τ wirkt nur auf W

A_α wirkt nur auf V

$$|\psi\rangle = \sum_{ij} c_{ij} |v_i\rangle \otimes |w_j\rangle$$

$$C|\psi\rangle = \sum_{ij} c_{ij} \sum_\tau \left(A_\alpha |v_i\rangle\right) \otimes \left(B_\alpha |w_j\rangle\right)$$

$$H_{\text{Ising}} = \sum_k \varepsilon_k \left(\gamma_k^+ \gamma_k - \tfrac{1}{2}\right)$$

$$\varepsilon_k = 2\sqrt{g^2 + J^2 - 2gJ \cdot \cos\left(\frac{2\pi k}{N}\right)}$$

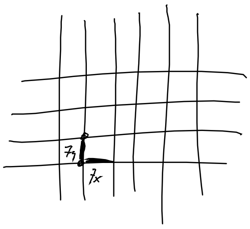
$$\rightsquigarrow E_0 = \sum_k \varepsilon_k \left(-\tfrac{1}{2}\right) \rightsquigarrow \frac{E_0}{N} \to \int_{-1/2}^{+1/2} \varepsilon(k)\, dk$$

kleinste Anregung $k = 1/2$ $\rightsquigarrow E_1 = E_0 + 2 \cdot \varepsilon_{1/2}$

Modell verhält sich nichtanalytisch bei $\underline{g = J}$ & $N \to \infty$

- auch bei $T = 0$ „Quantenphasenübergang"
- nichtanalyt. Verhalten ist in Observablen sichtbar.

Verbindung zum 2d Ising-Modell



$$Z_C = \operatorname{Tr}\left\{e^{-\beta H}\right\} = \operatorname{Tr}\left\{e^{+\beta g \sum \sigma_i^x + \beta \cdot J \sum \sigma_i^z \sigma_{i+1}^z}\right\}$$

$$= \sum_{\underline{z}} \langle z_1 \ldots z_N | e^{-\beta H} | z_1 \ldots z_N \rangle$$

$\sigma^z |0\rangle = +1 |0\rangle$

$\sigma^z |1\rangle = -1 |1\rangle$

$$\sigma_i^z |z_1 \ldots z_N\rangle = (-1)^{z_i} |z_1 \ldots z_N\rangle$$

$$|z_1\rangle \otimes |z_2\rangle \otimes \ldots \otimes |z_N\rangle = |\underline{z}\rangle = |z_1 \ldots z_N\rangle$$

Trotter-Formel

$$e^{A+B} = \lim_{L\to\infty}\left(e^{A/L}\, e^{B/L}\right)^L \qquad \sum_i |z_i\rangle\langle z_i| = \mathbb{1}$$

$$Z_C = \sum_{z_1}\lim_{L\to\infty}\langle z_1|\underbrace{\left(e^{+\beta H_0/L}\, e^{\beta H_1/L}\right)\cdot\ldots\cdot\left(e^{-\beta H_0/L}\, e^{-\beta H_1/L}\right)}_{L\text{ mal}}|z_1\rangle$$

- $\langle z_\alpha|\, e^{+\frac{\beta J}{L}\sum_i \sigma^z_i \sigma^z_{i+1}}\,|z_{\bar\alpha}\rangle = e^{\frac{\beta J}{L}\sum_i \sigma^i_\alpha \sigma^{i+1}_\alpha}\cdot\delta_{z_\alpha z_{\bar\alpha}}$
- $\langle z_\alpha|\, e^{\frac{\beta g}{L}\sum_i \sigma^x_i}\,|z_{\bar\alpha}\rangle = \ldots \qquad [\sigma^x_i, \sigma^x_j] = 0$

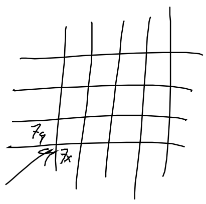
$$e^{\frac{\beta g}{L}\sigma^x} = \sum_{n=0}^{\infty}\left(\frac{\beta g}{L}\right)^n \frac{1}{n!}\left(\sigma^x\right)^n = \sum_{n=0}^{\infty}\left(\frac{\beta g}{L}\right)^{2n}\frac{1}{(2n)!}\cdot\mathbb{1} + \sum_{n=0}^{\infty}\left(\frac{\beta g}{L}\right)^{2n+1}\frac{1}{(2n+1)!}\cdot\sigma^x$$

$$= \cosh\left(\frac{\beta g}{L}\right)\cdot\mathbb{1} + \sinh\left(\frac{\beta g}{L}\right)\cdot\sigma^x = \begin{pmatrix}\cosh\left(\frac{\beta g}{L}\right) & \sinh\left(\frac{\beta g}{L}\right)\\ \sinh\left(\frac{\beta g}{L}\right) & \cosh\left(\frac{\beta g}{L}\right)\end{pmatrix}$$

$$\langle z^i_\alpha|\, e^{\frac{\beta g}{L}\sigma^x_i}\,|z^i_{\bar\alpha}\rangle = \cosh\left(\frac{\beta g}{L}\right)\cdot\delta_{z^i_\alpha z^i_{\bar\alpha}} + \sinh\left(\frac{\beta g}{L}\right)\left[1-\delta_{z^i_\alpha z^i_{\bar\alpha}}\right]$$

$$= \Lambda\cdot e^{\gamma\,\sigma^i_\alpha\sigma^i_{\bar\alpha}}$$

$$Z_C \longrightarrow \sum_{\sigma^i_\alpha\in\{-1,+1\}}\Lambda^{N\cdot L}\exp\Big\{\underbrace{\frac{\beta J}{L}}_{\bar\beta\cdot J_x}\sum_{i\alpha}\sigma^i_\alpha\sigma^{i+1}_\alpha + \underbrace{\gamma}_{\bar\beta\cdot J_y}\sum_{i\alpha}\sigma^i_\alpha\sigma^i_{\alpha+1}\Big\}$$



$x = i$
$y = \alpha$

$$J = \frac{L}{\beta}\cdot\bar\beta\cdot J_x$$

$$g = \frac{L}{\beta}\cdot\operatorname{arcoth}\left(e^{2\bar\beta\cdot J_y}\right)$$

krit. Punkt: $g = J \rightsquigarrow \coth\left(\bar\beta_{crit}\cdot J_x\right) = e^{2\bar\beta\cdot J_y}$

$J_x = J_y = J_{class}$

$$\beta_c\cdot J_{class} = \frac{1}{2}\ln\left(1+\sqrt{2}\right) \approx 0.441$$

Einschub: kollektive Spinmodelle (Drehimpulskopplung)

Wie beschreibt man Modelle mit kollektivem Spin

$$J^\alpha = \frac{1}{2}\sum_{n=1}^{N}\sigma^\alpha_n \qquad \alpha\in\{x,y,z\}$$

$$[\sigma^x_a, \sigma^y_b] = 2i\,\sigma^z_a\cdot\delta_{ab}$$

$$J^2 = (J^x)^2 + (J^y)^2 + (J^z)^2 \quad\Rightarrow\quad \boxed{[J^x, J^y] = i\cdot J^z}$$

$$\Rightarrow [J^2, J^z] = 0$$

suche Basis $|j, m\rangle$ in der J^2 & J^z diagonal sind

Bsp.: $N = 2$

$|00\rangle$

$|01\rangle \quad |10\rangle$ sind entartet

$|11\rangle$

$J^+ \uparrow \; |00\rangle \; \downarrow J^-$

$\frac{1}{\sqrt{2}}[|01\rangle + |10\rangle]$ $\qquad$ $\frac{1}{\sqrt{2}}[|01\rangle - |10\rangle]$

$J^+ \uparrow \; |11\rangle \; \downarrow J^-$

$$[J^z, J^\pm] \stackrel{!}{=} \pm J^\pm \quad \rightsquigarrow \quad \boxed{J^\pm = J^x \pm i J^y}$$

Leiter operatoren

$$\boxed{\begin{aligned} J^2 |j\,m\rangle &= j(j+1)\,|j\,m\rangle \\ J^z |j\,m\rangle &= m\,|j\,m\rangle \\ J^\pm |j\,m\rangle &= \underbrace{\sqrt{j(j+1) - m(m\pm 1)}}_{\text{Clebsch-Gordon-koeff.}}\,|j, m\pm 1\rangle \end{aligned}}$$

$$-j \le m \le +j$$

$$j \in \begin{cases} 0, 1, \dots \frac{N}{2} & \text{falls } N \text{ gerade} \\ \frac{1}{2}, \frac{3}{2}, \dots \frac{N}{2} & \text{falls } N \text{ ungerade} \end{cases}$$

Bsp.: $N = 2$

$|00\rangle \hat{=} |j=1, m=+1\rangle$

$\frac{1}{\sqrt{2}}[|01\rangle + |10\rangle] \hat{=} |j=1, m=0\rangle$ $\qquad$ $\frac{1}{\sqrt{2}}[|01\rangle - |10\rangle] \hat{=} |j=0, m=0\rangle$

$|11\rangle \hat{=} |j=1, m=-1\rangle$

$z_1 = 1 \quad z_2 = 1$

$N = 3$ $\qquad$ $j = 1/2$

$|000\rangle \hat{=} |j=\frac{3}{2}, m=+\frac{3}{2}\rangle$

$\frac{1}{\sqrt{3}}[|100\rangle + |010\rangle + |001\rangle] \hat{=} |j=\frac{3}{2}, m=+\frac{1}{2}\rangle$

$\frac{1}{\sqrt{3}}[|110\rangle + |011\rangle + |101\rangle] \hat{=} |j=\frac{3}{2}, m=-\frac{1}{2}\rangle$

$J^+ \uparrow$ $\quad$ $|111\rangle \hat{=} |j=\frac{3}{2}, m=-\frac{3}{2}\rangle$

$J^+ \quad J^-$

$j=3/2$ $j=5/2$ $j=7/2$

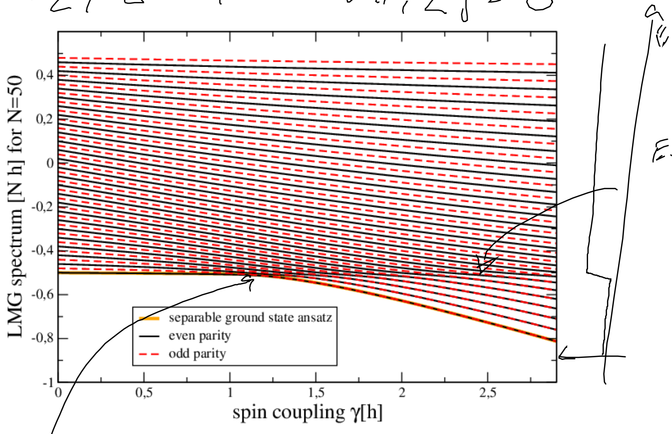
3.3.3. Lipkin-Meshkov-Glick-Modell

◦ historisch: Dynamik von Kernspins in ext. Feld

$$\hat H = -h\hat J^z - \frac{\gamma}{N}\left(\hat J^x\right)^2 \qquad [J^z, (J^x)^2] \neq 0$$

$$\Sigma = e^{-i\pi\frac{N}{2} + i\pi \hat J^z} \qquad [\hat H, \hat J^2] = 0$$

$$\rightsquigarrow \hat\Sigma \hat J^x \hat\Sigma = -J^x \qquad [H, \Sigma] = 0$$



ESQPT

lim $N\to\infty$ $\gamma = h$ QPT

Skizze der Diagonalisierung

$$\boxed{\begin{aligned} J^+ &= \sqrt{N\cdot 1 - a^\dagger a}\; a \\ J^- &= a^\dagger \sqrt{N\cdot 1 - a^\dagger a} \\ J^z &= \frac{N}{2}\cdot 1 - a^\dagger a \end{aligned}}$$

HP-Trafo

$$[J^x, J^y] = i J^z \Longleftrightarrow [a, a^\dagger] = 1$$

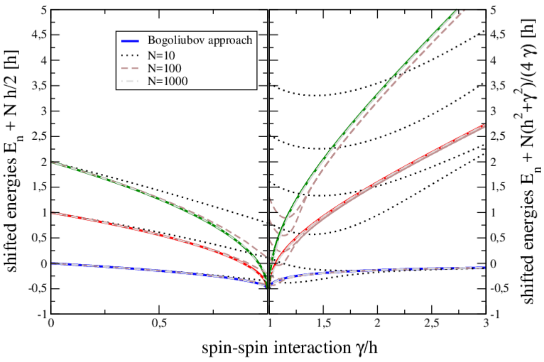
◦ $N\to\infty$ (TD-Limes): Wurzel entwickeln

◦ $\hat a = \hat b + \bar\alpha$, $\bar\alpha \in \mathbb{C}$ $\Rightarrow [b, b^\dagger] = 1$

◦ $b^\dagger b,\ b b^\dagger,\ b^2,\ (b^\dagger)^2$

$$b = \cosh(\varphi)\, c + \sinh(\varphi)\, c^\dagger \Rightarrow [c, c^\dagger] = 1, \quad \varphi \in \mathbb{R}$$

$$\Rightarrow \boxed{H = \pm E(h,\gamma)\, c^\dagger c + \alpha_1(h,\gamma) - \alpha_2(h,\gamma)\cdot N}$$



Bogoliubov approach
N=10
N=100
N=1000
shifted energies E_n + N h/2 [h]
shifted energies E_n + N(h^2+γ^2)/(4 γ) [h]
spin-spin interaction γ/h