Tut: Fr. $16^{00}-18^{00}$ $\Longrightarrow$ Do $16^{00}-18^{00}$

<u>Wdh</u>

- $\Delta\Phi = 0$ bei axialer Symmetrie

$$\Phi(r,\vartheta) = R(r)\,\Theta(\vartheta)$$
$$= \sum_{\ell=0}^{\infty}\left[A_\ell \cdot r^\ell + \frac{B_\ell}{r^{\ell+1}}\right] P_\ell(\cos\vartheta)$$

$A_\ell$ & $B_\ell$ folgen aus RB

- $\Delta\Phi = 0$ ohne ax. Symm.

$$\Phi(r,\vartheta,\varphi) = R(r)\, Y(\vartheta,\varphi)$$
$$r^2\Delta\, Y_{\ell m}(\vartheta,\varphi) = -\ell(\ell+1)\, Y_{\ell m}(\vartheta,\varphi) \qquad \delta_{\ell\ell'}\delta_{mm'} = \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\vartheta\, d\vartheta\, [\ldots]$$

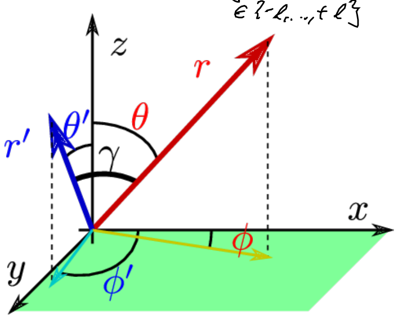
$$\Phi(r,\vartheta,\varphi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell}\underline{\left[a_{\ell m}\cdot r^\ell + \frac{b_{\ell m}}{r^{\ell+1}}\right]}\cdot Y_{\ell m}(\vartheta,\varphi) \qquad [\ldots] = Y^*_{\ell m}(\vartheta,\varphi)\, Y_{\ell' m'}(\vartheta,\varphi)$$

- $$\frac{1}{|\underline{r}-\underline{r}'|} = \sum_{\ell=0}^{\infty}\frac{1}{r_>}\left(\frac{r_<}{r_>}\right)^\ell P_\ell(\cos\gamma) \qquad \gamma = \sphericalangle(\underline{r},\underline{r}')$$

$$\frac{1}{|\underline{r}-\underline{r}'|} = \frac{1}{\sqrt{r^2+r'^2-2rr'\cos\gamma}}$$

$$\underline{r} = r\begin{pmatrix}\sin\vartheta\,\cos\varphi\\ \sin\vartheta\,\sin\varphi\\ \cos\vartheta\end{pmatrix} \qquad (\vartheta,\varphi,\vartheta',\varphi')$$

$$\frac{1}{|r-r'|} = \sum_{\ell\ell'=0}^{\infty}\sum_{\substack{m m'\\ m\in\{-\ell,\ldots,+\ell\}\\ m'\in\{-\ell',\ldots,+\ell'\}}}\underline{A_{\ell\ell' m m'}(r,r')}\cdot Y_{\ell m}(\vartheta,\varphi)\, Y^*_{\ell' m'}(\vartheta',\varphi')$$




$$\cos\gamma = \sin\vartheta\sin\vartheta'\left[\cos\varphi\cos\varphi' + \sin\varphi\sin\varphi'\right] + \cos\vartheta\cos\vartheta'$$

$$A_{\ell\ell' m m'}(r,r') = A_{\ell m}(r,r')\cdot\delta_{\ell\ell'}\delta_{mm'}$$

$$\left(\frac{1}{r}\frac{d^2}{dr^2}r - \frac{\ell(\ell+1)}{r^2}\right)A_\ell(r,r') = -\frac{4\pi}{r^2}\delta(r-r')$$

$$A_\ell(r,r') = \begin{cases} a_\ell\cdot r^\ell & : r<r' \\ b_\ell\cdot\frac{1}{r^{\ell+1}} & : r>r'\end{cases}$$

$$A_\ell(r'-\varepsilon,r') = A_\ell(r'+\varepsilon,r') \qquad \text{Stetigkeit}$$

i) $a_\ell \cdot r'^\ell = b_\ell \cdot \frac{1}{r'^{\ell+1}} \longrightarrow b_\ell = a_\ell \cdot (r')^{2\ell+1}$

ii)
$$\frac{d^2}{dr^2} r A_\ell(r,r') - \frac{\ell(\ell+1)}{r} A_\ell(r,r') = -\frac{4\pi}{r}\delta(r-r') \qquad \Big| \int_{r'-\varepsilon}^{r'+\varepsilon} [\ldots]\, dr \quad \lim \varepsilon \to 0$$

$$\left[\frac{d}{dr} r A_\ell(r,r')\right]_{r=r'-\varepsilon}^{r=r'+\varepsilon} + [0] = -\frac{4\pi}{r'}$$

$$a_\ell = \frac{4\pi}{2\ell+1}\frac{1}{(r')^{\ell+1}}$$

$$A_{\ell\ell',mm'}(r,r') = \frac{4\pi}{2\ell+1}\cdot\delta_{\ell\ell'}\delta_{mm'} \begin{cases} \frac{r^\ell}{r'^{\ell+1}} & : r<r' \\ \frac{r'^\ell}{r^{\ell+1}} & : r>r' \end{cases}$$

$$= \frac{4\pi}{2\ell+1}\delta_{\ell\ell'}\delta_{mm'}\cdot\frac{1}{r_>}\left(\frac{r_<}{r_>}\right)^\ell$$

$$\frac{1}{|r-r'|} = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell} \frac{4\pi}{2\ell+1}\frac{1}{r_>}\left(\frac{r_<}{r_>}\right)^\ell Y^*_{\ell m}(\vartheta',\varphi')\, Y_{\ell m}(\vartheta,\varphi)$$

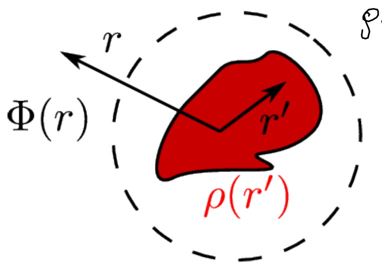
$$\boxed{P_\ell(\cos\gamma) = \frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{+\ell} Y^*_{\ell m}(\vartheta',\varphi')\, Y_{\ell m}(\vartheta,\varphi)}$$

$$\cos\gamma = \cos\vartheta\cos\vartheta' + \sin\vartheta\sin\vartheta'\cdot\cos(\varphi-\varphi')$$

Add.-Theorem der KFF

## 2.5. Multipolentwicklung

$\varrho(r>R)=0$




$$\Phi(r) = \int_V \frac{\varrho(r')}{|\underline{r} - \underline{r}'|}\, d^3r'$$

entwickeln für $|r| \gg |r'|$

### 2.5.1. sphärische Entw.

$$\Phi(r,\vartheta,\varphi) = \int \varrho(r')\,\frac{1}{r}\sum_{\ell=0}^{\infty}\left(\frac{r'}{r}\right)^\ell \cdot P_\ell(\cos\gamma)\, d^3r'$$

$$= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell} \frac{4\pi}{2\ell+1}\frac{1}{r^{\ell+1}}\, Y_{\ell m}(\vartheta,\varphi) \underbrace{\left[\int Y^*_{\ell m}(\vartheta',\varphi')\,\varrho(\underline{r}')(r')^\ell\, d^3r'\right]}_{q_{\ell m}}$$

$$q_{\ell m} = \int_0^\infty r'^2 dr' \int_0^{2\pi} d\varphi' \int_0^{\pi} \sin\vartheta'\, d\vartheta'\; \varrho(\underline{r}')(r')^\ell\, Y^*_{\ell m}(\vartheta',\varphi')$$

Multipolmomente

speziell $l = m = 0$ "Monopolmoment" (1)

$l=1,\ m \in \{-1,0,+1\}$ "Dipolmoment" (3)

$l=2\ \ m\in\{-2,\dots,+2\}$ "Quadrupol-..." (5)

a.) $q_{00} = \frac{1}{\sqrt{4\pi}} \int \varrho(\underline{r}')\, d^3r' = \frac{q}{\sqrt{4\pi}}$ Monopolmoment $\hat{=}$ Gesamtladung

b.) Dipol $\quad q_{1,-1} = \int Y^*_{1,-1}(\vartheta',\varphi')\varrho(\underline{r}')|\underline{r}'|\, d^3r'$

$$= \sqrt{\frac{3}{4\pi}} \int \sin\vartheta' [\cos\varphi' + i\sin\varphi'] \cdot \varrho(\underline{r}') \cdot r'\, d^3r'$$

$$= \sqrt{\frac{3}{4\pi}} \int (x' + iy')\varrho(r')\, d^3r'$$

$$q_{1,+1} = \sqrt{\frac{3}{4\pi}} \int (-x' + iy')\, \varrho(r')\, d^3r'$$

$$q_{1,0} = \sqrt{\frac{3}{4\pi}} \int z' \varrho(r')\, d^3r'$$

$$\Phi(r,\vartheta,\varphi) = \underbrace{\frac{q}{r}}_{\substack{q \\ \text{dominiert im Fernfeld}}} + \frac{\underline{p}\cdot\underline{r}}{r^3} + \dots \mathcal{O}\left(\frac{1}{r^3}\right)$$

$$\underline{p} = \int \underline{r}\, \varrho(\underline{r}')\, d^3r'$$

## 2.5.2. Kartesische Multipol-EW

$$\Phi(\underline{r}) = \int \frac{\varrho(\underline{r}')}{|r - r'|}\, d^3r' = \int \varrho(r') \left[ \frac{1}{|\underline{r}|} + \sum_{i=1}^{3} \frac{x_i x_i'}{r^3} + \frac{1}{2!} \sum_{i,j=1}^{3} \frac{3x_i x_j - \delta_{ij} r^2}{r^5} x_i' x_j' + \dots \right] d^3r'$$

$$= \frac{q}{r} + \frac{\underline{r}\cdot\underline{p}}{r^3} + \frac{1}{2!} \sum_{i,j=1}^{3} \frac{3x_i x_j - r^2\delta_{ij}}{r^5} \underbrace{\int \varrho(r') x_i' x_j'\, d^3r'}_{3\times 3 \text{ mit 6 Komponenten}} + \dots$$

$$\underline{p} = \int \underline{r}'\, \varrho(\underline{r}')\, d^3r' = \int \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \varrho(r')\, d^3r'$$

letzten Term

$$\frac{1}{3!}\sum_{ij}\frac{3x_ix_j - r^2\delta_{ij}}{r^5}\left[\int \rho(r')\left[\underline{3x_i'x_j' - r'^2\delta_{ij}} + \underline{r'^2\delta_{ij}}\right]d^3r'\right]$$

nutze $\sum_{ij}\left(3x_ix_j - r^2\delta_{ij}\right)\cdot\delta_{ij} = \sum_{i=1}^{3}\left(3x_i^2 - r^2\right) = 3r^2 - 3r^2 = 0$

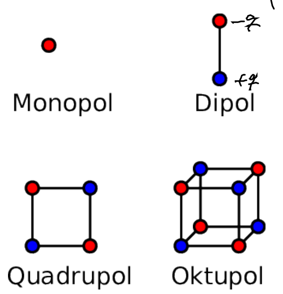
$$\Rightarrow = \frac{1}{3!}\sum_{ij}\frac{3x_ix_j - r^2\delta_{ij}}{r^5}\, Q_{ij}$$

kart. Quadrupolmoment $\boxed{Q_{ij} = \int \rho(r')\left[3x_i'x_j' - r'^2\delta_{ij}\right]d^3r'}$

$Tr\{Q_{ij}\} = \sum_{i=1}^{3} Q_{ii} = 0$ "$Q_{ij} = Q_{ji}$ 6 unabh. + Spur → 5 übrig"

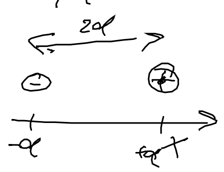
$$P = \begin{pmatrix}P_x\\P_y\\P_z\end{pmatrix} = \sqrt{\frac{4\pi}{3}}\begin{pmatrix}\frac{1}{\sqrt{2}}\left(q_{1,-1} - q_{1,+1}\right)\\ \frac{1}{i\sqrt{2}}\left(q_{1,-1} + q_{1,+1}\right)\\ q_{1,0}\end{pmatrix}$$

höhere Terme




Reine Multipole

Dipol $\rho = q\left(\delta(x-d) - \delta(x+d)\right)\delta(y)\delta(z)$




$$\underline{P} = \int \begin{pmatrix}x'\\y'\\z'\end{pmatrix}\rho(x',y',z')\,d^3r' = \begin{pmatrix}2\cdot q\cdot d\\0\\0\end{pmatrix}$$

Dipol ≙ Ladung mal Abstand

## 2.6. WW-Energie im äußeren Feld

vgl. 1.8.

$$W = \sum_i q_i \cdot \Phi_e(r_i) \rightarrow \int_V \rho(\underline{r}) \cdot \Phi_e(\underline{r})\, d^3r$$

$\rho$ am Ursprung lokalisiert

$\Phi_e(\underline{r})$: entwickle in Reihe um $r=0$

$E_e$ wird von einer weit entfernten Ladung erzeugt $\rightarrow$ $\forall r \in V$ gilt: $\underline{\nabla} \underline{E}_e = 0$

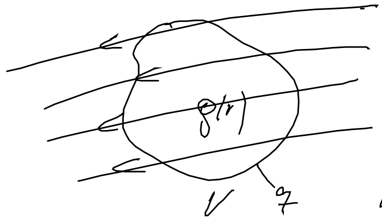
$$\Phi_e = \Phi_e(0) + \underline{r}\cdot(\nabla \Phi_e)\Big|_{r=0} + \frac{1}{2}\sum_{i,j=1}^{3} \frac{\partial^2 \Phi_e}{\partial x_i \partial x_j} \cdot x_i x_j + \ldots$$

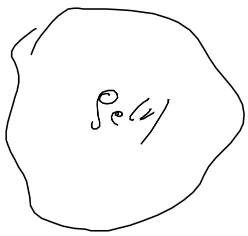
$$= \Phi_e(0) - \underline{r}\cdot(\underline{E}_e(0)) - \frac{1}{2}\sum_{i,j=1}^{3} \frac{\partial E_i}{\partial x_j}\Big|_{x_i=0} \cdot x_i x_j + \ldots$$

$$0 = \frac{1}{6}\cdot r^2 (\nabla \cdot E_e)$$

$=$

$$E_e = -\nabla \int \frac{\rho_e(\underline{r}')}{|\underline{r}-\underline{r}'|} d^3r'$$




$$\nabla E_e = 4\pi \rho_e$$

$$\forall r \in V: \nabla E_e = 4\pi \rho_e\Big|_V = 0$$

$$\Phi(r) = \Phi_e(0) - \underline{r}\cdot \underline{E}_e(0) - \frac{1}{6}\sum_{i,j=1}^{3}\left(3x_i x_j - r^2\delta_{ij}\right)\frac{\partial E_{e,j}}{\partial x_i}\Big|_0$$

in W ersetzen

$$W = q\cdot\Phi_e(0) - \underline{p}\cdot\underline{E}_e(0) - \frac{1}{6}\sum_{i,j=1}^{3} Q_{ij}\frac{\partial E_{e,j}}{\partial x_i}\Big|_0 + \ldots$$

Monopol mit Potential
Dipol mit Feldstärke
Quadrupol mit Abl. der Feldst.

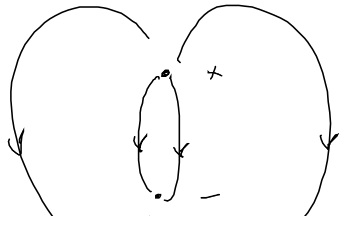
<u>Beispiel</u> Pot. eines (reinen) Dipols am Ursprung

$$\Phi_1(r) = \underbrace{\frac{\underline{p}_1\cdot\underline{r}}{r^3}}_{\text{Dipolmoment}} = \frac{p_1^x\cdot x + p_1^y\cdot y + p_1^z\cdot z}{[x^2+y^2+z^2]^{3/2}}$$

$$E_1(r) = -\nabla\Phi_1(r) = -\left[\frac{p_1}{r^3} - \frac{3}{2}\frac{(\underline{p}_1\underline{r})\cdot 2\cdot r}{r^5}\right]$$

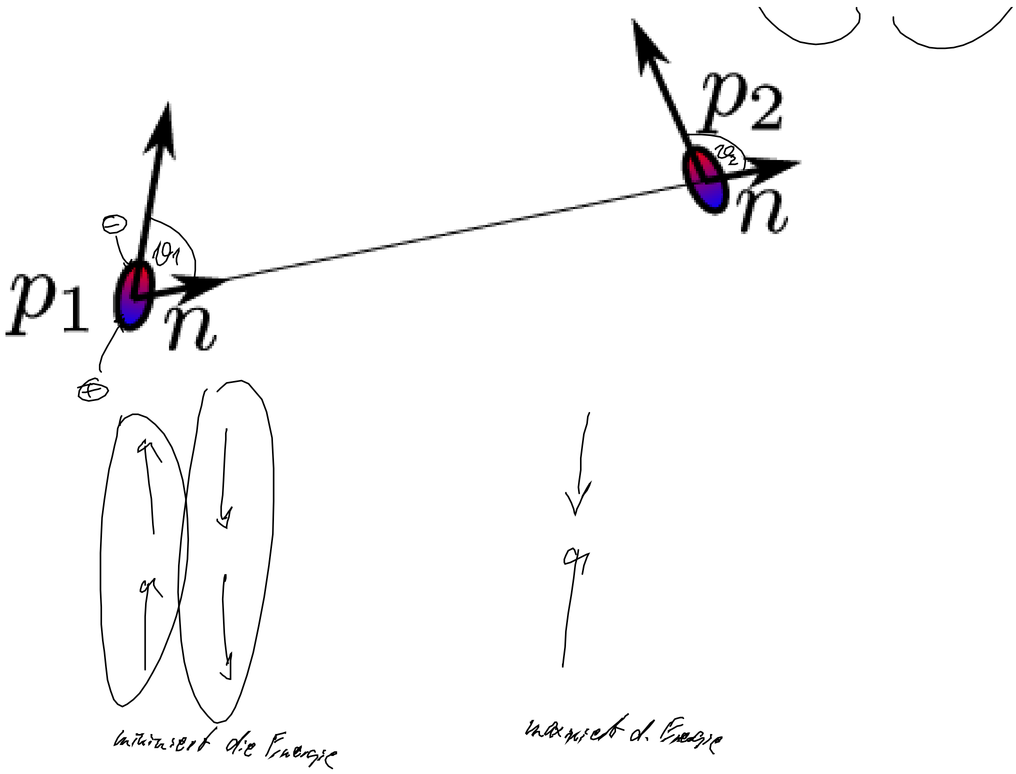
$$\underline{n} = \frac{\underline{r}}{|\underline{r}|}$$

$$= -\frac{p_1 - 3(\underline{n}\cdot\underline{p}_1)\cdot\underline{n}}{r^3} \mathrel{\hat{=}} E_{ext}$$




WW-Energie zwischen 2 Dipolen

$$W_{12} = -p_2\cdot E_1 = +\frac{p_1\cdot p_2 - 3(n\cdot p_1)(n\cdot p_2)}{r^3}$$


p1
n
p2
n
minimiert die Energie
maximiert d. Energie