

Cumulants and Moments

ν -th moment $M_\nu = \langle x^\nu \rangle$

$$\begin{aligned} \text{Moment-generating function } Z(\alpha) &= \langle e^{\alpha x} \rangle = \\ &= \sum_{\nu=0}^{\infty} \frac{\alpha^\nu}{\nu!} M_\nu \quad ; \quad \frac{\partial^\nu}{\partial \alpha^\nu} Z(\alpha) \Big|_{\alpha=0} = M_\nu \end{aligned}$$

The moment-generating function is named so because it can be used to find the moments of the distribution. The knowledge about all moments is equivalent to knowing the probability density function.

$d = is$: inverse Fourier transform

$$Z(is) = \int dx \, \varphi(x) e^{isx}$$

$$\varphi(x) = \frac{1}{2\pi} \int ds \, Z(is) e^{-isx}$$

Generalization to d random variables
(d dimensions)

$$M_{\nu_1 \nu_2 \dots \nu_d} = \langle x_1^{\nu_1} x_2^{\nu_2} \dots x_d^{\nu_d} \rangle$$

Moments of order $\nu = \nu_1 + \nu_2 + \nu_3 + \dots + \nu_d$

Moment generating function

$$Z(\underline{\alpha}) = \langle e^{\underline{\alpha} \cdot \underline{x}} \rangle = \sum_{\nu_1 \dots \nu_d} \frac{\alpha_1^{\nu_1} \dots \alpha_d^{\nu_d}}{\nu_1! \dots \nu_d!} M_{\nu_1 \dots \nu_d}$$

$$\underline{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_d)$$

$$\underline{x} = (x_1, x_2, \dots, x_d)$$

Cumulant

Cumulant $C_{j_1 \dots j_d} = \langle x_1^{j_1} x_2^{j_2} \dots x_d^{j_d} \rangle_c$ defined by

cumulant-generating function

$$\Gamma(\underline{d}) = \ln \langle e^{\underline{d} \cdot \underline{x}} \rangle = \sum_{j_1 \dots j_d} \frac{d_1^{j_1} \dots d_d^{j_d}}{j_1! \dots j_d!} C_{j_1 \dots j_d}$$

Properties: cumulants additive for uncorrelated random variables (not valid for moments!)

$$\langle (x_1 + x_2)^2 \rangle = \langle x_1^2 \rangle + \langle x_2^2 \rangle + 2 \langle x_1 x_2 \rangle$$

Proof: Let x_1, x_2 be uncorrelated $\underline{d} = (d_1, d_2)$

$$\begin{aligned} \Rightarrow Z(\underline{d}) &= \langle e^{\underline{d} \cdot \underline{x}} \rangle = \int dx_1 dx_2 \rho(x_1) \rho(x_2) e^{d_1 x_1} e^{d_2 x_2} = \\ &= \langle e^{d_1 x_1} \rangle \langle e^{d_2 x_2} \rangle \end{aligned}$$

$$\Rightarrow \Gamma(\underline{d}) = \ln Z(\underline{d}) = \ln \langle e^{d_1 x_1} \rangle + \ln \langle e^{d_2 x_2} \rangle \stackrel{!}{=} \Gamma(d_1) + \Gamma(d_2)$$

$$\begin{aligned} d_1 = d_2 = d: \Gamma(d, d) &= \ln \langle e^{\overbrace{d(x_1 + x_2)}^x} \rangle = \\ &= \sum_j \frac{d^j}{j!} \langle (x_1 + x_2)^j \rangle_c = \\ &= \sum_j \frac{d^j}{j!} \langle x_1^j \rangle_c + \sum_j \frac{d^j}{j!} \langle x_2^j \rangle_c \end{aligned}$$

$$\Rightarrow \langle (x_1 + x_2)^j \rangle_c = \langle x_1^j \rangle_c + \langle x_2^j \rangle_c$$

□