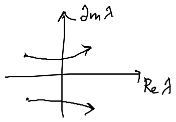
Lecture 17 summary

3.4 Coherence resonance in non-excitable systems

18.12.19 → Projects in Tutorial
06.01.20 → no lecture
08.01.20 → first lecture in 2020

3.4.1 Hopf bifurcations

HB – appearance / disappearance of the limit cycle (periodic solution)
related to the change of stability of a fixed point:
a pair of complex conjugate eigenvalues (linearization around fixed point)
crosses the imaginary axis in the complex plane



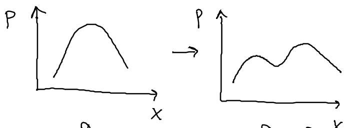
Hopf bifurcation

supercritical
$$\boxed{\begin{aligned}\dot r &= \mu r - r^3\\ \dot\theta &= \omega + b r^2\end{aligned}}$$

subcritical
$$\boxed{\begin{aligned}\dot r &= \mu r + r^3 - r^5\\ \dot\theta &= \omega + b r^2\end{aligned}}$$

3.4.2 Stochastic bifurcations

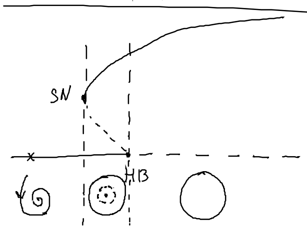
P-bifurcations

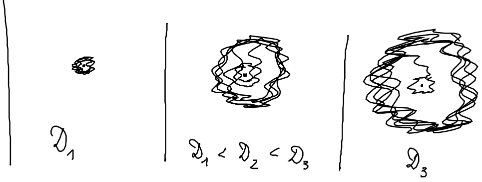


D_1 $\quad$ $D_2 \neq D_1$

Example of P-bifurcation

subcritical Hopf / deterministic	stochastic





D_1 $\quad$ $D_1 < D_2 < D_3$ $\quad$ D_3

→ stochastic P-bifurcations

Stochastic bifurcation → P-bifurcation, D-bifurcation

P-bifurcation $\quad$ D-bifurcation

D-bifurcation (dynamical) – change of stability of trajectories belonging to a certain set with a given measure (invariant). For example, a sign change of one of the Lyapunov exponents.

3.4.3 Coherence resonance in Stuart-Landau oscillator

Ushakov et al. Phys. Rev. Lett 95, 2005

Stuart-Landau oscillator

$$\dot{z} = -i\omega_0 z + z F(z) + \sqrt{2D}\,\xi(t)$$

z is complex variable
$z = x + iy$

$F(z) = a_1 - |z|^2$ supercritical HB

$F(z) = a_2 + |z|^2 - |z|^4$ subcritical HB

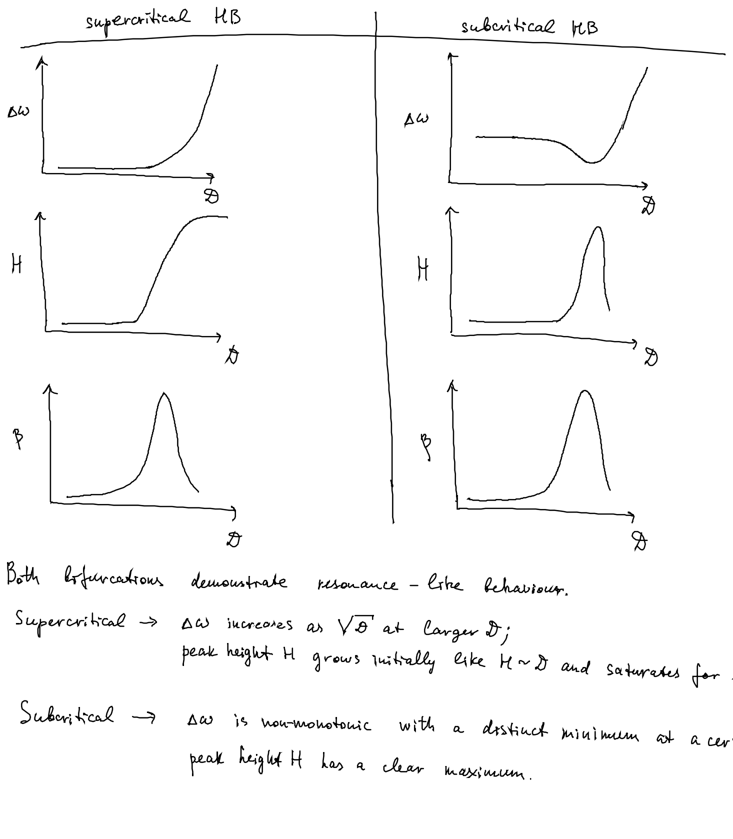
ω_0 – eigenfrequency

$r = |z| = \sqrt{x^2 + y^2}$

Measure of CR → signal-to-noise ratio (SNR)

$$\beta = \frac{H}{\Delta\omega / \omega_p}$$

CR occurs in a strict sense only in the subcritical case.



Both bifurcations demonstrate resonance-like behaviour.

Supercritical → $\Delta\omega$ increases as $\sqrt{D}$ at larger D;
peak height H grows initially like $H \sim D$ and saturates for stronger noise

Subcritical → $\Delta\omega$ is non-monotonic with a distinct minimum at a certain noise level
peak height H has a clear maximum.

Supercritical case $\rightarrow$ the increase of SNR is produced by the spectral peak height H, that is by an increase of the oscillation amplitude. The width $\Delta\omega$ is initially only weakly affected, but increases steeply for stronger noise, weakening the coherence. The resonance in β $\rightarrow$ due to competition between the growth of H and $\Delta\omega$.

Subcritical case $\rightarrow$ $\Delta\omega$ itself demonstrates a minimum $\rightarrow$ noise improves, indeed the temporal coherence of oscillations.

3.4.4. Duffing – van der Pol oscillator

Zakharova et al. Phys. Rev. E 2010

$$\ddot{x} - (\varepsilon + x^2 - x^4)\dot{x} + x + \beta x^3 = \sqrt{2D}\, n(t) \quad , \beta \geq 0$$

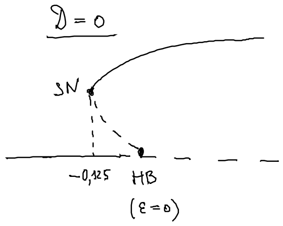
$n(t)$ is Gaussian white noise : $\langle n(t)\, n(t+\tau)\rangle = \delta(\tau)$

$\langle n(t)\rangle = 0$

D is noise intensity

β defines anisochronicity of oscillations : $\beta = 0 \rightarrow$ system is isochronous $\rightarrow$ the frequency of oscillations does not depend on the amplitude.

$D = 0$



$-\frac{1}{8} < \varepsilon < 0$ (between SN and HB)

bistability : stable focus + stable LC

Analytical approach: we assume that D is small $\rightarrow$ fluctuations of the amplitude and phase are "slow" stochastic processes.

$\Rightarrow$ they remain unchanged during the period $T_0 = 2\pi$ $(\omega_0 = 1)$

We change variables:

$$x(t) = a(t)\cos[t + \varphi(t)], \quad \dot{x}(t) = -a(t)\sin[t + \varphi(t)]$$

$a(t)$ – instantaneous amplitude

$\varphi(t)$ – instantaneous phase

We substitute new variables into the equation of D.-van der Pol oscillator and average the equations over the period of oscillations.

[see details of the method in R.L. Stratonovich, Selected Topics in the Theory of Random Noise 1963, vol. 1 and 2]

We obtain stochastic equations for the slow random variables:

$$\boxed{\begin{aligned} \dot{a} &= \left(\frac{\varepsilon}{2} + \frac{a^2}{8} - \frac{a^4}{16}\right) a + \frac{D}{2a} + \sqrt{D}\, n_1(t), \\ \dot{\varphi} &= \frac{3\beta}{8} a^2 + \frac{\sqrt{D}}{a} n_2(t), \end{aligned}}$$

$n_1(t)$ and $n_2(t)$ are independent sources of Gaussian white noise

From these equations we can derive the amplitude of stable limit cycle for $D=0$: $a_0 = \sqrt{1 + \sqrt{1+8\varepsilon}}$

An important observation $\rightarrow$ $\dot{a}$ does not depend on φ $\Rightarrow$

$\Rightarrow$ $p(a,t)$ rather than $p(a, \varphi, t)$

joint probability density for φ and a

$$\dot{a} = \left(\frac{\varepsilon}{2} + \frac{a^2}{8} - \frac{a^4}{16}\right) a + \frac{D}{2a} + \sqrt{D}\, n_1(t)$$